Numerical Methods for Differential Equations

Fundamental Concepts for Scientific and Engineering Applications

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Chapter 1

Characteristics and Boundary Conditions for Partial Differential Equations

1.0 INTRODUCTION

Education in mathematical principles and methods typically starts with discrete systems and progresses toward continuous systems. For example, a child’s initial exposure to real numbers deals exclusively with integers (and at first, only positive integers). This then develops toward study of fractions, decimals, and finally irrational numbers. Thus understanding of real numbers begins with a few locations on the real number axis and proceeds to include all points on the axis. In learning about calculation of the slope of a curve in \((x, y)\) space, students initially consider straight lines for which the slope is given by the discrete expression \(\Delta y/\Delta x\). Understanding of the concept of slope is completed when one is able to calculate the slope at any point on an arbitrary curve using the continuous expression \(dy/dx\). When studying science or engineering, mathematical models for physical processes are presented in terms of algebraic expressions (e.g., \(F = ma\)) before they are seen in terms of differential equations [e.g., \(F = d(mv)/dt\)].

The study of numerical methods for the solution of differential equations is complex, in part, because it runs counter to the usual pattern of moving from the discrete to the continuous. Numerical methods, in brief, replace a differential description of a physical process or system with an approximate discrete analog and then solve this analog to represent the solution of the differential system. The challenge to the numerical modeler
is twofold: (1) to develop analogs that approximate the differential system with a known and tolerable degree of accuracy, and (2) to employ solution algorithms that are efficient and usable within the constraints of the cost and availability of computing power.

Effective and creative numerical solution of differential equations requires that the modeler (1) have an understanding of the physical processes occurring, (2) be able to describe those processes through differential equations, (3) be able to formulate a discrete analog to the differential system, and (4) be able to solve that analog on the computer. The main focus of this book is on the latter two of these four indispensable components of modeling, as indicated in Figure 1.1. However, because the physical behavior of a system under study influences the selection of a numerical technique, the techniques in this book are more insightfully understood when considered in the context of particular physical phenomena described by differential equations rather than from a purely mathematical viewpoint.

1.1 BEHAVIOR OF PHYSICAL SYSTEMS

Differential equations in science and engineering are attempts to describe the actual behavior of physical systems. In determining an appropriate mathematical representation, one must identify the scale, in both time and space, at which the system is to be modeled.
Sec. 1.1 Behavior of Physical Systems

For example, if the behavior of a container of gas subjected to temperature variations is to be modeled, different phenomena could be studied, each at a different scale. In general, as the time or space scale is made larger, the equations describing the system become simpler; but less information is obtained. For the container of gas, phenomena may be described at the molecular, the continuum, or the global scale. At the molecular level, collisions of gas molecules with each other and with the walls of the container must be modeled. At the continuum level, wherein molecular structure and gaps between particles are neglected, temperature and pressure are variables that may be defined at every point in the system. At the global scale, only average properties for the entire system (averaged in time and/or space) can be described. As the scale increases, either spatially or temporally, the ability to describe smaller-scale phenomena is compromised; but the complexity of interactions among equations, of the equation form, or of the boundary conditions usually decreases. The material presented in this book is intended to provide the reader with exposure to and facility with numerical procedures for solution of differential equations that describe phenomena occurring at the continuum scale or larger.

For the equations under study here, at most four independent variables will appear: three spatial dimensions and time. When the problem considered is time invariant, it is called a steady-state system. Time-dependent problems are said to be transient. The special case of transient but periodic (repeated) behavior is called dynamic steady state. The dimensionality of a problem refers to the number of spatial dimensions over which variation is considered. Thus a two-dimensional transient problem would depend on time and two spatial directions. In developing numerical solution procedures or algorithms appropriate for a problem of interest, the domain of the solution must be taken into account such that the portion of time and space which is to be considered is clearly defined. For example, suppose that one is interested in describing the changing pressure field in a groundwater aquifer during a period of constant pumping from a well. The response of the aquifer to the pumping may be considered, rather generally, to depend on the vertical direction, the two lateral directions from the well, and time. Thus the domain of the solution is three spatial dimensions and time. One could develop a general model that allows for variation of pressure in each of these variables. However, this degree of detail is often unnecessary in obtaining a reasonably accurate description of the pressure field. For example, if the flow is primarily horizontal toward the well, the pressure distribution in the vertical tends toward hydrostatic and the vertical dimension can be eliminated. The pressure field can thus be modeled by solving a partial differential equation in time and the two lateral dimensions. If, in addition, the aquifer is homogeneous and isotropic, such that flow to the well may be considered to be radially symmetric, the pressure field may be modeled using an equation that depends only on time and radial distance from the well. If the aquifer is very large in lateral extent but one is only interested in the short-term response of the pressure field, an adequate model may be formulated by describing the changing pressure field in the vicinity of the well but holding it fixed at its initial value at some relatively large distance from the well.

To model any physical system with reasonable accuracy, one must simulate a domain in time and space that includes the region where significant variation of the
variables of interest occurs. Supplementary conditions along the edge of the domain must also be prescribed a priori in order to complete the model. Qualitatively, this can be understood in the context of groundwater flow by realizing that the pressure field in the aquifer at any particular time would depend on the field at an earlier time as well as the physical boundaries of the domain (i.e., whether the boundary is a river, impermeable rock, or a slightly permeable formation). Alternatively, modeling of the evolution of a temperature field in a solid body would require that the locations of heat sources and the magnitudes of these sources be known in addition to the initial temperature field and the conditions along the boundary. The mathematical exposition of boundary conditions is provided in Section 1.3.

Before proceeding to definitions pertaining to properties of differential equations, three important behaviors of these equations will be discussed in relation to physical systems and the domain of solution. The purpose of this discussion is to motivate classification of differential equation types from the perspective of properties of the solution rather than from simply the coefficients and terms in the equation. This discussion is less precise than the mathematical section that follows but does provide some insight into important considerations in selecting needed features of numerical solution algorithms.

The first system to be considered is the rectangular plate depicted in Figure 1.2a subjected to an arbitrary temperature distribution around its boundary. Assume, for convenience, that this plate is insulated so that no energy transfer occurs in the direction normal to the plate. A time-invariant temperature profile for this plate will result if the boundary temperature is held fixed for a long period of time. The domain of solution of

![Figure 1.2](image-url)
Sec. 1.1 Behavior of Physical Systems

the differential equation describing this system is the portion of $x$-$y$ space that the plate covers. Now, if the arbitrary temperature profile imposed at the boundary is changed at just one point on the boundary, the steady-state temperature profile within the plate will change at every point within the domain. For example, if the boundary temperature profile is changed only at point $A$ in Figure 1.2b, the temperature will be altered in the shaded portion of the domain (i.e., every point in the domain except along the edge where the original temperature is specified). A numerical approximation to the differential equation describing the problem must take into account the fact that each point on the boundary affects the solution at every point within the domain.

As a second example, consider the one-dimensional rod in Figure 1.3a. Let the problem of interest be the temporal development of the temperature profile along the

Figure 1.3  (a) One-dimensional rod of length $x_2-x_1$. (b) Solution domain for the temperature distribution in the rod. (c) Region of solution domain influenced by the temperature specified at point $A$. (d) Region of solution domain influenced by the temperature specified at point $B$. 

(a) Rectangular plate
(b) Temperature distribution at initial time and boundary. (b) Change in temperature at point $A$ for a uniform temperature on entire plate.
length of the rod and let variations in temperature over the cross section be neglected. The domain of solution is the portion of $x$-$t$ space depicted in Figure 1.3b which spans the length of the rod and extends infinitely into time beginning from $t^0$, the time at which a disturbance to an initial temperature profile in the rod is applied. Suppose that at time $t^0$, a steady-state temperature profile is disturbed by changing the temperatures at the ends of the rods to some new values. If the ends of the rod are held at these new temperatures, the temperature profile will evolve to some final distribution. The differential equation usually selected to model this system is marked by two notable attributes: (1) it predicts that the final, steady-state temperature distribution occurs only after an infinite amount of time; and (2) it predicts that a change in temperature at either end of the rod immediately affects the entire rod. Both of these attributes are mathematical approximations to reality. However, for most systems, the predicted temperature field is negligibly different from that obtained by laboratory experiment.

For this model of heat transfer in a rod, it is interesting to note how changes in the specified temperature along the boundaries of the domain alter the solution. If two initial temperature profiles are considered that differ at just one point, $A$ in Figure 1.3c, the temperature distributions at a later time will be different within the entire domain, the shaded region in the figure (except, of course, when the ultimate steady-state solution is reached).

Next consider two similar temperature evolutions in the rod where initial conditions at $t^0$ and end conditions at $x_1$ and $x_2$ are identical. Now at point $B$ indicated in Figure 1.3d, time $t = t^b$ and $x = x_B$, assume that the temperature of the end of the rod is changed for one of the tests. Thus at all points within the domain of solution for time greater than $t^b$, the shaded portion of the figure, the temperature fields of the two solutions will be different. However, temperature fields will be the same in the domain for time less than $t^b$. This is, perhaps, not surprising, because an event occurring at one time can affect only the future, not the past. Of particular note in consideration of this example is the line separating the changed and unchanged portions of the domain in Figure 1.3d. A curve of this sort did not arise in the rectangular plate example nor in the current system when only the initial conditions were modified. The feature of a condition along the edge of the domain affecting portions of the domain differently must be accounted for properly in any numerical approximation to the differential equation. Solution techniques must be designed to deal with domains that are finite as well as those that are infinite in one or more variables.

As a third example, consider a perfectly elastic (nondamping) string that is stretched between $x = x_1$ and $x = x_2$. If the string is given an initial displacement and released from rest (i.e., plucked) at $t = t^0$, the displacement will propagate back and forth along the string as depicted in Figure 1.4a. The domain of solution is the finite section of $x$ occupied by the string and the infinite portion of the time axis that begins at $t = t^0$, as shown in Figure 1.4b. Note that the disturbance will propagate in both the positive and negative $x$-directions and does not instantaneously affect the entire spatial domain as in the preceding example. If the initial displacement considered is modified at point $A$, this modification will cause the wave to differ from the original wave along the curves emanating from point A in Figure 1.4c and in the shaded region. Note that there are
two lines emerging from point A which are important. In the unshaded portion of the domain, the solution is unaffected by changes at point A. This observation contrasts with the previous examples of temperature changes in the rod, where a change of the initial condition influenced the solution at every point in space, and where the change at the \( x \)-boundary influenced all points in the spatial domain for time later than the time the disturbance was imposed. In addition, while the impact of a boundary condition perturbation decreases with distance in the domain from the perturbation for the heat conduction examples, perturbations at the boundary persist in the solution for the string displacements and move back and forth along the string with time. A numerical procedure appropriate for simulating the latter phenomenon must be able to preserve the shape of the wave by propagating it at the correct velocity and maintaining its magnitude.
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The selection or development of appropriate computational algorithms for the solution of differential equations is aided by a priori knowledge of the behavior of the system under study. If an analytical solution to a differential equation can be obtained, there is no need to obtain a numerical solution (although a comparison between analytical and numerical solutions is a useful step in assessing the performance of a numerical scheme). When an analytical solution is not available, selection of the numerical scheme, discretization of the domain, and application of boundary conditions must all be done judiciously to obtain a good numerical solution. For some problems, this information is well estimated because of the modeler’s knowledge of the system under study. In other cases, this information is obtained by successive refinements of the procedures employed based on solutions obtained. Thus numerical models provide insight into the behavior of physical systems as well as quantifying this behavior in systems that are qualitatively well understood.

Before proceeding with the development of numerical methods for the solution of differential equations, it is appropriate to provide a few definitions and conventions which will facilitate the discussion that follows.

1.2 DEFINITIONS AND EQUATION PROPERTIES

Differential equations may be classified as either ordinary differential equations (ODEs), in which only one independent variable appears, or as partial differential equations (PDEs), in which more than one independent variable appears. Within the scope of the current study, there are four main independent variables: time and the three spatial dimensions. The order of a differential equation is the order of the highest derivative that appears. The degree of the equation is the greatest power to which the highest-order derivative is raised. Therefore,

\[
\frac{d^2 u}{dt^2} + u \frac{du}{dt} - (\frac{du}{dt})^3 = t^3
\]  

(1.2.1)

is a second-order, first-degree ODE;

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0
\]

(1.2.2)

is a second-order, first-degree PDE; and

\[
\left( \frac{\partial u}{\partial x} \right)^2 + x \frac{\partial^3 u}{\partial x^3} = f(x, t)
\]

(1.2.3)

is a first-order, second-degree PDE. Differential equations are sometimes rearranged to reduce the explicit appearance of order in the equation. For example, if

\[
q = u - \frac{\partial u}{\partial x}
\]

(1.2.4a)

equation (1.2.2) becomes

\[
\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0
\]

(1.2.4b)
Sec. 1.2 Definitions and Equation Properties

Despite the fact that no second derivative appears in equations (1.2.4), this coupled set of first-order equations is second order because combination of these equations into a single form yields a second-order form. Note that a higher-order equation can always be written as a set of first-order equations, but the converse is not true.

A homogeneous differential equation does not contain a term involving only the independent variables. Therefore, equation (1.2.1) is nonhomogeneous because of the \( t^3 \) term, equation (1.2.2) is homogeneous, and equation (1.2.3) is nonhomogeneous unless the arbitrary function \( f(x, t) \) is equal to zero.

Another important concept in classifying differential equations is linearity. An \( n \)-th order differential equation is said to be linear if the dependent variable and its derivatives appear only to the zero or first degree in an equation and no products of the dependent variable and its derivatives with other derivatives of the variable appear. (This follows from the general definition of linearity given in Appendix A.) Therefore, equation (1.2.2) is linear, whereas equations (1.2.1) and (1.2.3) are not linear because \( du/dt \) and \( \partial u/\partial x \) are raised to the third and second degree in these equations, respectively. Note that a linear equation may have coefficients of the dependent variable or its derivatives that depend on the independent variables, so that

\[
x^3 \frac{\partial^2 u}{\partial x \partial t} + e^{-t} \frac{\partial u}{\partial x} + u = 0
\]

is a linear PDE. The coefficients in this equation—\( x^3 \), \( e^{-t} \), and 1—do not depend on \( u \) or its derivatives. A differential equation that is not linear is nonlinear, and the special case of a nonlinear equation in which the highest-order derivatives occur linearly is said to be quasilinear. For example,

\[
\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = 0
\]

is a nonlinear equation and

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} = 0
\]

is quasilinear. For convenience, subscripts will be used to indicate derivatives at some places in the text (primarily in this chapter) such that equation (1.2.7) is alternatively expressed as

\[
u_t + \nu u_x - \nu_{xx} = 0
\]

Differential equations of importance in science and engineering range from relatively simple linear, homogeneous, first-order ordinary differential equations to highly complex nonlinear, high-order, multidimensional partial differential equations. Although this range of equation types is interesting, for purposes of developing numerical solution algorithms, the behavior of the solution is more important. In Section 1.1, three different types of behavior were discussed, each of which depended on how perturbations influenced the solution domain. In the following section, methods for classifying equations by type are developed and the implications of type in specifying appropriate boundary conditions are explored.
1.3 CHARACTERISTICS AND BOUNDARY CONDITIONS

The solution of a differential equation requires that appropriate boundary conditions, from both a mathematical and a physical viewpoint, be specified. An ODE for which conditions are imposed at only one end of the domain is called an initial value problem. However, if conditions are imposed at both ends of the domain, the ODE is called a boundary value problem. On the other hand, a PDE may be an initial value problem in one independent variable and a boundary value problem in another. The determination of whether a PDE or system of PDEs forms an initial value problem or a boundary value problem is aided by introducing the concept of characteristics. For the case of a PDE with two independent variables, real characteristics, if they exist, are curves in the plane of the independent variables along which information propagates. When a differential equation has three independent variables, the real characteristics are surfaces; and in higher dimensions, the real characteristics are hypersurfaces.

Perturbations in the boundary conditions influence regions interior to the domain by propagating along characteristics. In some problems, specified boundary conditions may cause the solution to be discontinuous. The locations of these discontinuities can be determined by following them as they propagate into the system along characteristics. Also of importance is the observation that because a characteristic may be a locus of discontinuity, knowledge of the solution only along a characteristic does not necessarily provide knowledge of the solution elsewhere in the domain. For example, given the solution on a characteristic, one could not obtain the solution at a point near the characteristic using the differential equation and a Taylor series expansion because the discontinuity along the characteristic would cause some of the needed derivatives to be indeterminate. This observation implies that boundary conditions should not be specified along characteristics since information cannot propagate from the characteristics into the domain of solution. In fact, as will be shown subsequently, curves or surfaces along which boundary conditions are specified must cut across characteristics. Specification of appropriate boundary conditions for a differential equation is based on an examination of the characteristics and how information on these characteristics influences the solution.

1.3.1 First-Order Partial Differential Equations (Two Independent Variables)

The simplest first-order partial differential equation is

\[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0 \]  \hspace{1cm} (1.3.1)

where \( v \) is positive and constant. The restriction to constant \( v \) is for convenience, and relaxation of this restriction does not dramatically alter the conclusions that follow. This equation describes, for example, convection in a tube of a dissolved chemical concentration front, moving with uniform velocity \( v \). For purposes of compactness of notation, manipulations will be performed on equation (1.3.1) expressed using the subscript notation as

\[ u_t + vu_x = 0 \]  \hspace{1cm} (1.3.2)
Sec. 1.3 Characteristics and Boundary Conditions

The solution to this equation for \( u \) is sought in the domain \( 0 < x < L \) and \( 0 < t < T \). This differential equation only describes how \( u \) varies with time and space; it says nothing about the actual value of \( u \). Thus to determine \( u \) completely, some auxiliary conditions, also referred to as boundary conditions or initial conditions, must be established. These conditions correspond to specification of the value of \( u \) along a curve, or curves, in the \( x \)-\( t \) plane. As mentioned previously, and as will be discussed further, such a curve must intersect the characteristics and may not be a characteristic. It is therefore important to know the characteristic family for a differential equation prior to specifying a boundary condition.

For this example, the characteristics may be obtained easily. However, for consistency and as an introduction to the procedure needed for more complex systems, the characteristics will be derived formally. Recall that a characteristic is a curve along which a singularity may propagate. For the present case, define an arbitrary continuous function in \( x \)-\( t \) space as \( \varphi(x, t) \). When \( \varphi \) is a constant, a relationship between \( x \) and \( t \) exists and a curve in \( x \)-\( t \) space is defined. Now assume that \( u(x, t) \) of equation (1.3.2) is defined along some curve \( \varphi(x, t) \) equal to a constant. Therefore, in general, \( u_x \) can be related to \( u_t \) through the derivative of \( u \) along the curve.

If one is positioned at a point in \( x \)-\( t \) space and moves an infinitesimal distance \( dx \) in \( x \) and \( dt \) in \( t \), the change in \( \varphi \) will be

\[
d\varphi = \varphi_t dt + \varphi_x dx
\]  

(1.3.3)

Now if the change in position is restricted such that \( \varphi \) has the same value at the beginning and end of the step, then \( d\varphi = 0 \) and the movement is along the curve of constant \( \varphi(x, t) \). This type of motion is particularly important and will be indicated by denoting \( dt \) along the curve as \( D_t \), and \( dx \) along the curve as \( D_x \). (Note that \( D\varphi = 0 \) by definition since the capital \( D \) indicates movement along a curve of constant \( \varphi \).) Therefore, equation (1.3.3) becomes

\[
D\varphi = 0 = \varphi_t D_t + \varphi_x D_x
\]  

(1.3.4)

or

\[
\frac{D_t}{D_x} = -\frac{\varphi_x}{\varphi_t}
\]  

(1.3.5)

Now consider the change in \( u \) encountered by moving an infinitesimal distance in space:

\[
du = u_t dt + u_x dx
\]  

(1.3.6)

If this motion is constrained to be on a curve of constant \( \varphi \), then

\[
Du = u_t D_t + u_x D_x
\]  

(1.3.7)

where \( Du \) is not necessarily zero because \( u \) may vary along the curve \( \varphi(x, t) = \text{constant} \). Equation (1.3.7) may be solved for \( u_x \) as

\[
u_x = \frac{Du}{D_x} - u_t \frac{D_t}{D_x}
\]  

(1.3.8)
But from equation (1.3.5), \(\frac{Dt}{Dx} = -\frac{\varphi_x}{\varphi_t}\), so that equation (1.3.8) becomes

\[
u_t = \frac{Du}{Dx} + u_t \frac{\varphi_x}{\varphi_t}
\]  \hspace{1cm} (1.3.9)

Note that this equation relates \(u_t\) to \(u_x\) and a derivative of \(u\) along the curve. In addition, the governing equation (1.3.2) relates \(u_t\) to \(u_x\) at every point in space. Thus substitution of equation (1.3.9) into equation (1.3.2) eliminates \(u_x\) and yields

\[
\left(1 + v \frac{\varphi_x}{\varphi_t}\right) u_t = -v \frac{Du}{Dx}
\]  \hspace{1cm} (1.3.10)

This equation indicates that if \(u\) is specified along a curve \(\varphi(x, t) = \text{constant}\), \(u_t\) may be calculated in terms of a derivative along the curve unless \(1 + \nu(\varphi_x/\varphi_t) = 0\). In matrix form, equation (1.3.10) may be written

\[
\begin{bmatrix}
1 + v \frac{\varphi_x}{\varphi_t}
\end{bmatrix}
\begin{bmatrix}
u_t
\end{bmatrix} =
\begin{bmatrix}
-v \frac{Du}{Dx}
\end{bmatrix}
\]  \hspace{1cm} (1.3.11)

This \(1 \times 1\) matrix equation has been written to illustrate formally the fact that the ability to solve for \(u_t\) requires that the determinant of the coefficient matrix be nonzero. Thus, if the curve on which \(u\) is specified is such that

\[
1 + v \frac{\varphi_x}{\varphi_t} = 0
\]  \hspace{1cm} (1.3.12)

or

\[
\varphi_t + vu_x = 0
\]  \hspace{1cm} (1.3.13)

then \(u_t\) cannot be determined from values of \(u\) on the curve. This indeterminacy condition serves to define the characteristic curves. The characteristic curves may be solved for using equation (1.3.13) such that

\[
\varphi = \varphi \left((x - x_0) - v(t - t^0)\right)
\]  \hspace{1cm} (1.3.14)

where \(x_0\) and \(t^0\) are arbitrary constants. Note that because \(\varphi\) will change only when \(x - vt\) changes, a different characteristic curve is determined for each value of \(x - vt\). Therefore, the family of characteristic curves can be denoted by

\[
x - vt = \text{constant}
\]  \hspace{1cm} (1.3.15a)

or more simply as

\[
(x - x_0) - v(t - t^0) = 0
\]  \hspace{1cm} (1.3.15b)

where selection of different values of the parameters \(x_0\) and \(t^0\) provides different curves in the family of characteristics. By convention, equation (1.3.15b) is referred to as the characteristic equation for equation (1.3.2) because it gives rise to a set of characteristic curves for different values of the constants. Although an infinite number of curves exists, each conforms to the single constraint (1.3.15a). Equation (1.3.2) is thus said to have one characteristic.

Selection of various values for \(x_0\) and \(t^0\) and illustration of \(t\) versus \(x\) leads to the family of characteristic curves in Figure 1.5a. Complete knowledge of \(u\) along
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(1.3.8) becomes

(1.3.9)

These are characteristics of the curve. In addition, analytic techniques are possible in terms of the characteristic of the curve. Thus substitution

(1.3.10)

In each case, if $v = constant$, $u_t$ may be found.

(1.3.11)

Thus, the fact that the ability of the characteristic matrix be nonzero. Thus,

(1.3.12)

(1.3.13)

Indeterminacy condition arises in some cases may be solved for

(1.3.14)

Thus $u$ will change only when

(1.3.15a)

(1.3.15b)

This provides different curves (1.3.15b) is referred to as the characteristic in a set of characteristic curves. For each number of curves exists, (1.3.15b) is thus said to have

(1.3.15c)

Thus the graph of $t$ versus $x$ leads to

(1.3.15d)

knowledge of $u$ along

\[
\begin{align*}
\text{Figure 1.5} & \quad \text{(a) Characteristics of equation (1.3.2) as determined by differential (1.3.15b). (b) Region of influence of conditions specified on } x \text{-axis, } 0 < x < L, \text{ in solving equation (1.3.2). (c) Region of influence of conditions specified on } t \text{-axis, } 0 < t < T, \text{ in solving equation (1.3.2). (d) Combined region of influence for conditions specified on } x \text{-axis, } 0 < x < L, \text{ and } t \text{-axis, } 0 < t < T.}
\end{align*}
\]
one of these curves, together with the differential equation, is insufficient information for calculation of \( u \) throughout the domain of interest, \( 0 < t < T \) and \( 0 < x < L \) (see Problem 1.2 for further exposition of this point). In contrast, if \( u \) were known, for example, along the curve \( \varphi = x/L + t/T = 1 \), a curve that intersects all the characteristics for this problem, it would be possible to solve for \( u \) everywhere in the domain of interest. Although such a specification is mathematically possible, it is unlikely that an experiment would be performed in which \( u \) along the curve \( x/L + t/T = 1 \) is controlled. Therefore, physically realistic as well as mathematically correct boundary conditions will be considered.

Typically, one would expect to be able to specify an initial state for the system, that is, the concentration at all positions along the tube at \( t = 0 \) as indicated by the wavy line along the \( x \)-axis in Figure 1.5b. Specification of only this condition would allow the concentration to be solved for only in the shaded “region of influence” in Figure 1.5b. Concentration above this region and within the problem domain would be affected by more than just the initial condition. Specification of only the inflow concentration (i.e., the concentration along \( x = 0 \) for time \( 0 < t < T \)), as indicated by the wavy line along the \( t \)-axis in Figure 1.5c, is sufficient to determine the concentration only in the shaded region of influence indicated in the figure. Thus adequate auxiliary conditions for the problem defined by equation (1.3.1) are initial condition and inlet condition specifications along the portion of the lines \( x = 0 \) and \( t = 0 \), as indicated in Figure 1.5d.

As an important physical consideration, one should note that time and space are inherently different. Perturbations at a spatial location may, in general, influence the state of the system in all spatial directions around the point. However, disturbances occurring at a particular time are incapable of affecting the system at an earlier time but can only influence the state at a subsequent time. Thus, in the present problem, where \( v \) is positive, the appropriate location for specifying the spatial boundary condition is at \( x = 0 \) because information along this line propagates into the spatial domain as time increases. If the flow were reversed such that \( v \) became negative, characteristics would have a negative slope in the \( x-t \) plane; and the appropriate spatial location for specifying a boundary condition would be at \( x = L \). With negative \( v \), it is information at \( x = L \) that propagates into the spatial domain with increasing time.

The partial differential equation considered in this section is classified as an initial value problem in both time and space because conditions on the boundary of the \( x-t \) domain are specified at only one \( t \) and one \( x \) location. Higher-order PDEs may be either initial value problems, boundary value problems (with conditions specified at both boundaries of the domain), or mixed problems (initial value problems with respect to one or more independent variables, and boundary value problems with respect to other independent variables). The appropriate classification and boundary condition requirements can be obtained from an analysis of characteristics.

### 1.3.2 Second-Order Partial Differential Equations (Two Independent Variables)

A fairly general second-order differential equation with independent variables \( x \) and \( t \) may be written as
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\[ Au_{xx} + 2Bu_{xt} + Cu_{tt} + Du_t + Eu_x = R \quad (1.3.16) \]

where \( A, B, C, D, \) and \( E \) may be functions of \( x, t, u, u_x, u_t, \) and \( u, \) while \( R \) may be a function of \( x, t, \) and \( u. \) This equation is quasilinear since it is nonlinear and first degree.

For consistency of presentation with Section 1.3.1, the single second-order equation will be converted to a system of first-order equations by first defining the variables:

\[ f = u_t \quad (1.3.17) \]

\[ g = u_x \quad (1.3.18) \]

Substitution of these expressions into equation (1.3.16) in order to replace second derivatives of \( u \) with first derivatives of \( f \) and \( g \) yields

\[ Ag_{xx} + Bg_t + Bf_x + Cf_t + Du_t + Eu_x = R \quad (1.3.19) \]

Furthermore, differentiation of equation (1.3.17) with respect to \( x \) and of equation (1.3.18) with respect to \( t \) demonstrates that

\[ f_x = g_t \quad (1.3.20) \]

These last four equations can be examined to determine whether or not curves exist along which specification of \( u, f, \) and \( g \) is inadequate to determine the solution of (1.3.16) in a region of interest. If so, these are the characteristic curves.

The curves to be studied lie in the \( x-t \) plane and are indicated in general as \( \varphi(x, t) = constant. \) The question that will be explicitly addressed here is: If \( u, f, \) and \( g \) are specified on a curve \( \varphi(x, t), \) can \( u_t, f_x, \) and \( g_t \) be determined using equations (1.3.17) through (1.3.20)? Of course, if the curve is \( \varphi = x \) (i.e., a curve parallel to the \( t \)-axis) the partial derivatives with respect to \( t \) of \( u, f, \) and \( g \) are simply their respective derivatives along the curve. In that case, a more interesting question is: Can \( u_t, f_x, \) and \( g_x \) be determined using equations (1.3.17) through (1.3.20)? Fortunately, if one is careful with the algebraic manipulations, both questions can be answered by doing only one analysis. Arbitrarily, the first question is addressed here, with the second question left to the reader as Problem 1.3.

In the same manner as the previous analysis, if one is located at a point in \( x-t \) space where \( \varphi \) has a particular value and moves an arbitrary infinitesimal distance \( dx \) and an arbitrary infinitesimal distance \( dt, \) the change in \( \varphi \) will be

\[ d\varphi = \varphi_t dt + \varphi_x dx \quad (1.3.21) \]

If, however, the movement is constrained such that the value of \( \varphi \) does not change, then equation (1.3.21) becomes

\[ D\varphi = 0 = \varphi_t dt + \varphi_x dx \quad (1.3.22) \]

where the notational change from lowercase \( d \) to uppercase \( D \) is made to emphasize that movement is on a \( \varphi = constant \) curve. Rearrangement of equation (1.3.22) yields

\[ \frac{Dt}{Dx} = -\frac{\varphi_x}{\varphi_t} \quad (1.3.23) \]
The change in $u$ experienced by moving an infinitesimal distance from an initial location $x, t$ is

$$du = u_t dt + u_x dx$$  \hspace{1cm} (1.3.24)

If this movement is also constrained such that it occurs along a curve of constant $\varphi$, then

$$Du = u_t D_t + u_x D_x$$  \hspace{1cm} (1.3.25)

or

$$\frac{Du}{Dx} = u_t \frac{Dt}{Dx} + u_x$$  \hspace{1cm} (1.3.26)

Substitution of equation (1.3.23) into (1.3.26) and rearrangement yields

$$u_x = \frac{Du}{Dx} + u_t \frac{\varphi_x}{\varphi_t}$$ \hspace{1cm} (1.3.27a)

By analogous arguments using $f$ and $g$ instead of $u$ in equations (1.3.24) through (1.3.27a), the following equations are obtained

$$f_x = \frac{Df}{Dx} + f_t \frac{\varphi_x}{\varphi_t}$$ \hspace{1cm} (1.3.27b)

and

$$g_x = \frac{Dg}{Dx} + g_t \frac{\varphi_x}{\varphi_t}$$ \hspace{1cm} (1.3.27c)

Equations (1.3.27) are now used to eliminate $u_x$, $f_x$, and $g_x$ from equation (1.3.19), to obtain

$$A \frac{Dg}{Dx} + Ag_t \frac{\varphi_x}{\varphi_t} + Bg_t + B \frac{Df}{Dx} + Bf_t \frac{\varphi_x}{\varphi_t} + Cf_t + Du_t + E \frac{Du}{Dx} + Eu_t \frac{\varphi_x}{\varphi_t} = R$$ \hspace{1cm} (1.3.28)

Because $u$, $f$, and $g$ are prescribed along a specified curve $\varphi$, the derivatives along the curve may be calculated and the only unknowns in this last equation are $u_t$, $f_t$, and $g_t$. After moving the known quantities to the right side and collecting terms on the left, equation (1.3.28) is rewritten as

$$\left(D + E \frac{\varphi_x}{\varphi_t}\right) u_t + \left(C + B \frac{\varphi_x}{\varphi_t}\right) f_t + \left(B + A \frac{\varphi_x}{\varphi_t}\right) g_t = R - D \frac{Du}{Dx} - B \frac{Df}{Dx} - A \frac{Dg}{Dx}$$ \hspace{1cm} (1.3.29)

Now recall definition (1.3.17),

$$u_t = f$$ \hspace{1cm} (1.3.30)

Next use equation (1.3.20) to replace $f_x$ with $g_t$ in equation (1.3.27b) and obtain

$$- \frac{\varphi_x}{\varphi_t} f_t + g_t = \frac{Df}{Dx}$$ \hspace{1cm} (1.3.31)
These last three equations may be combined as the matrix equation

$$
\begin{bmatrix}
D + E \frac{\varphi_x}{\varphi_t} & C + B \frac{\varphi_x}{\varphi_t} & B + A \frac{\varphi_x}{\varphi_t} \\
1 & 0 & 0 \\
0 & -\frac{\varphi_x}{\varphi_t} & 1
\end{bmatrix}
\begin{bmatrix}
u_t \\
f_t \\
g_t
\end{bmatrix}
= 
\begin{bmatrix}
R - E \frac{Du}{Dx} - B \frac{Df}{Dx} - A \frac{Dg}{Dx} \\
0 \\
0
\end{bmatrix}
$$

Because the matrix is square, \(u_t, f_t,\) and \(g_t\) may be solved for using the known right-side vector and the known coefficients in the matrix unless the determinant of the matrix is zero. A zero determinant indicates that \(\varphi\) is a characteristic curve since the ability to solve for \(u_t, f_t,\) and \(g_t\) is precluded in this case. Calculation of the determinant and imposition of the constraint that it equal to zero yields

$$- \left( C + B \frac{\varphi_x}{\varphi_t} \right) - \left( B + A \frac{\varphi_x}{\varphi_t} \right) \frac{\varphi_x}{\varphi_t} = 0$$

or, after multiplication by \(\varphi_t^2\) and rearrangement,

$$A\varphi_x^2 + 2B\varphi_x \varphi_t + C\varphi_t^2 = 0$$

It is interesting to note that this expression, which is satisfied if \(\varphi\) is a characteristic, is independent of \(D\) and \(E\), the coefficients of the first derivatives. This quadratic equation has two roots and may be solved for \(\varphi_x/\varphi_t\) (or \(\varphi_t/\varphi_x\)) by the standard procedure for a quadratic equation. Solution for \(\varphi_x/\varphi_t\) yields

$$\frac{\varphi_x}{\varphi_t} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

while solution for \(\varphi_t/\varphi_x\) yields

$$\frac{\varphi_t}{\varphi_x} = \frac{-B \pm \sqrt{B^2 - AC}}{C}$$

[Note that equation (1.3.35a) with the \(+\) \((-\) sign is the inverse of (1.3.35b) with the \(-\) \((+\) sign.] From these expressions it can be seen that if \(B^2 - AC > 0\), there are two unique real characteristics; if \(B^2 - AC = 0\), there is only one unique real characteristic; and if \(B^2 - AC < 0\), both characteristics are imaginary. From analytic geometry it is known that, with some restrictions on \(D, E,\) and \(F,\) an equation of the form

$$A\varphi_x^2 + 2B\varphi_t + C\varphi_t^2 = Dx + Et + F$$

describes a hyperbola if \(B^2 - AC > 0,\) a parabola if \(B^2 - AC = 0,\) and an ellipse if \(B^2 - AC < 0.\) Therefore, by extension of the terminology, the general second-order differential equation given in (1.3.16) may be classified as hyperbolic, parabolic, or elliptic as follows:
If $B^2 - AC > 0$: the equation is hyperbolic, has two unique real characteristics, and has no complex characteristics.

If $B^2 - AC = 0$: the equation is parabolic, has one unique real characteristic, and has no complex characteristics.

If $B^2 - AC < 0$: the equation is elliptic, has no real characteristics, but has two unique complex characteristics. These characteristics are complex conjugates.

Although the general classification scheme discussed above is comprehensive, some particular cases are especially interesting. Often, when solving a second-order differential equation, some of the second derivatives are absent in that one or two of the coefficients $A$, $B$, and $C$ become zero in various regions of the solution domain. These cases have been tabulated and information concerning the characteristics has been compiled in Table 1.1. Note that in most cases, only the ratio of derivatives, $\varphi_x/\varphi_t$, has been presented; but for those cases where $\varphi_x$ or $\varphi_t$ equals zero, the actual expression for the characteristic is provided. Recall that along a characteristic, $\varphi$ is constant. Thus the characteristic $t - t^0 = 0$ is in fact a family of straight lines parallel to the $x$-axis that take on different $t$ values when different values of the arbitrary parameter $t^0$ are selected.

**TABLE 1.1 CHARACTERISTICS FOR EQUATION (1.3.16)**

<table>
<thead>
<tr>
<th>$A$</th>
<th>$B$</th>
<th>$C$</th>
<th>Characteristics</th>
</tr>
</thead>
</table>
| $\neq 0$ | $\neq 0$ | $\neq 0$ | 1. $\varphi_x/\varphi_t = (-B + \sqrt{B^2 - AC})/A$  
2. $\varphi_x/\varphi_t = (-B - \sqrt{B^2 - AC})/A$ |
| $\neq 0$ | $\neq 0$ | 0 | 1. $\varphi_x = 0$ such that $t - t^0 = 0$  
2. $\varphi_x/\varphi_t = -2B/A$ |
| $\neq 0$ | 0 | $\neq 0$ | 1. $\varphi_x/\varphi_t = \sqrt{-C/A}$  
2. $\varphi_x/\varphi_t = -\sqrt{-C/A}$ |
| 0 | $\neq 0$ | $\neq 0$ | 1. $\varphi_t = 0$ such that $x - x_0 = 0$  
2. $\varphi_x/\varphi_t = -C/2B$ |
| $\neq 0$ | 0 | 0 | 1. $\varphi_x = 0$ such that $t - t^0 = 0$  
2. Double root (see secondary characteristic) |
| 0 | $\neq 0$ | 0 | 1. $\varphi_x = 0$ such that $x - x_0 = 0$  
2. $\varphi_x = 0$ such that $t - t^0 = 0$ |
| 0 | 0 | $\neq 0$ | 1. $\varphi_t = 0$ such that $x - x_0 = 0$  
2. Double root (see secondary characteristic) |
| 0 | 0 | 0 | First-order equation (see secondary characteristic) |

One additional case presented in Table 1.1 that is appropriate for discussion at this time is the case where $A$, $B$, and $C$ are all zero. This is a degenerate case for which the second-order differential equation becomes first order. However, note that when $A$, $B$, and $C$ are all zero, equation (1.3.32) becomes
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\[
\begin{bmatrix}
D + E \frac{\varphi_x}{\varphi_t} & 0 & 0 \\
1 & 0 & 0 \\
0 & -\frac{\varphi_x}{\varphi_t} & 1
\end{bmatrix}
\begin{bmatrix}
u_t \\
f_t \\
g_t
\end{bmatrix} =
\begin{bmatrix}
R - E \frac{Du}{Dx} \\
f \\
Df \frac{Df}{Dx}
\end{bmatrix}
\] (1.3.37)

Because the equation has degenerated to a first-order form, the characteristic analysis can also degenerate to the analysis for a first-order problem. For this first-order analysis, further restrict \( D \) and \( E \) such that they do not depend on \( u_t \) or \( u_x \), and the equation is quasilinear. Now the question of interest is: If \( u \) is specified on a curve \( \varphi(x, t) \), can \( u_t \) be calculated using information provided by the differential equation? Thus equation (1.3.37) is reduced to the expression

\[
\left[ D + E \frac{\varphi_x}{\varphi_t} \right] [u_t] = \left[ R - E \frac{Du}{Dx} \right]
\] (1.3.38)

Here \( u_t \) cannot be calculated if the determinant of the matrix is zero, or

\[
D + E \frac{\varphi_x}{\varphi_t} = 0
\] (1.3.39)

Information concerning these \( \varphi \) curves for various values of \( D \) and \( E \) is presented in Table 1.2. Note that when \( A, B, \) and \( C \) are not all zero, these curves are not characteristics for the second-order differential equation. Nevertheless, they still provide some information concerning the behavior of the general second-order problem, as will be demonstrated subsequently. Therefore, these curves are important. To set these curves apart from the characteristics of the problem, they are referred to here as secondary characteristics. These secondary characteristics become the primary characteristics if \( A, B, \) and \( C \) are zero everywhere in the domain such that the differential equation is first-order.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\( D \) & \( E \) & Secondary characteristics \\
\hline
\( \neq 0 \) & \( \neq 0 \) & \( \varphi_x/\varphi_t = -D/E \) \\
\( \neq 0 \) & \( 0 \) & \( \varphi_t = 0 \) such that \( x - x_0 = 0 \) \\
\( 0 \) & \( \neq 0 \) & \( \varphi_x = 0 \) such that \( t - t_0 = 0 \) \\
\( 0 \) & \( 0 \) & None \\
\hline
\end{tabular}
\caption{Secondary Characteristics for Equation (1.3.16)}
\end{table}

The mathematical manipulations of these sections have led to information concerning the existence, in real space, of characteristic curves along which discontinuities may propagate. These curves are also extremely useful in determining requirements on boundary conditions for a problem. In the next section, typical model problems from the three classifications of second-order partial differential equations in two independent variables—elliptic, parabolic, and hyperbolic—will be used to illustrate boundary condition requirements.
1.3.3 Example Second-Order Partial Differential Equations
(Two Independent Variables)

The first example to be treated in this section is the two-dimensional Laplace equation

\[ T_{xx} + T_{yy} = 0 \]  \hspace{1cm} (1.3.40)

This equation, also known as the steady-state heat or diffusion equation, describes, for example, the steady-state temperature profile in a plate where

- \( T \) is the temperature, and
- \( x, y \) are Cartesian coordinate directions.

Let the domain of interest be the rectangular region described by \( 0 < x < L \) and \( 0 < y < H \). With reference to the standard equation (1.3.16), \( A = C = 1 \) and \( B = D = E = R = 0 \). This corresponds to the third row in Table 1.1. Furthermore,

\[ B^2 - AC = -1 < 0 \]  \hspace{1cm} (1.3.41)

and equation (1.3.40) is elliptic with no real characteristics. Thus there are no curves emanating from the boundary along which disturbances propagate. A change in conditions at any point on the boundary alters the solution throughout the domain. From physical reasoning, this is not surprising. If the temperature of a plate is changed at any point on the boundary, one would expect to have a different steady-state temperature field.

Boundary conditions for differential equations are restricted to specification of some combination of the dependent variable and derivatives of this variable along the edge of the domain. The derivatives specified must be of lower order than the order of the governing equation. Specification of the value of the dependent variable is referred to as a first-type or Dirichlet boundary condition. The specification of the derivative of the dependent variable normal to the boundary, the normal derivative, is referred to as a second-type or Neumann boundary condition. Finally, specification of a linear combination of the dependent variable and its normal gradient along the boundary is a third-type or Robin boundary condition. Note that where a Robin condition is specified, neither the variable nor its normal gradient is known independently.

For an elliptic problem, one boundary condition must be specified at each point on the boundary. If only the gradient is specified at every point, a unique solution may not be obtainable. On physical grounds, this can be understood by referring to the heat equation (1.3.40) and recognizing that the gradient normal to the boundary is proportional to the heat flux (by Fourier's law). Thus the gradients could be specified so that heat entering the system along one portion of the boundary is precisely balanced by heat leaving along the remainder of the boundary. Although this is an equilibrium situation and the gradients within the domain can be obtained at every point, no information as to the actual temperature of the plate is provided. Alternatively, one could specify fluxes such that, for example, the heat entering the system is greater than that leaving. This would mean that the temperature could not be at a steady-state value and the equation would be inconsistent with the physical problem. To overcome these problems, the temperature must be specified for at least one point on the boundary; and either the temperature, the
characteristics and boundary conditions

Temperature gradient normal to the boundary, or a Robin condition must be specified at every other point. (In fact, in the present case, a mix of Robin conditions and Neumann conditions along the boundary will provide a unique temperature distribution. However, for some elliptic problems, a unique solution is not obtained without specification of the dependent variable for at least one point on the boundary.) The appropriate boundary conditions for the elliptic problem discussed here are depicted in Figure 1.6. The elliptic problem is a boundary value problem because boundary conditions must be specified at every point on a closed boundary. Problem 1.4 explores the question of nonuniqueness for the case of only Neumann conditions specified along the boundary.

As a second example, consider the equation describing the one-dimensional conduction of heat in a rod:

\[ T_t - \kappa T_{xx} = 0 \]  

where

- \( T \) is temperature,
- \( \kappa \) is the thermal diffusivity,
- \( t \) is time, and
- \( x \) is the coordinate along the axis of the rod.

Consider the case where the rod is aligned such that it extends from 0 to \( L \) on the \( x \)-axis, and the time evolution of the temperature distribution in the rod is to be studied for
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0 < t \leq t_{\text{max}}. Equation (1.3.42) corresponds to equation (1.3.16), with u replaced by T and A = -\kappa, B = 0, C = 0, D = 1, E = 0, and R = 0. Therefore, equation (1.3.42) corresponds to the fifth row in Table 1.1,

\[ B^2 - AC = 0 \]  

(1.3.43)

and the equation is parabolic. Equation (1.3.34) provides the equation for the characteristic

\[ -\kappa \varphi_x^2 = 0 \]  

(1.3.44)

This quadratic equation has two roots, \( \varphi_x = 0 \), but because they are identical, only one unique characteristic is defined:

\[ \varphi = \varphi(t - t^0) \]  

(1.3.45)

Because \( \varphi \) is constant along a characteristic, the functional form of the dependence of \( \varphi \) on \( t - t^0 \) is irrelevant and the characteristic curve family is selected as

\[ t - t^0 = 0 \]  

(1.3.46)

A plot of the characteristics is given in Figure 1.7.

![Figure 1.7 Primary characteristics of parabolic equation (1.3.42) in region 0 < x < L and t > 0 as determined by equation (1.3.46).](image)

Because the characteristics in this figure are parallel to the x-axis, the effects of disturbances at either end of the rod will propagate instantaneously to a point on the interior of the rod. Thus, based on Figure 1.7, one might expect that, for example, if both ends of the rod were set at 100°C, the rod would instantaneously achieve a uniform temperature of 100°C. There is nothing in the analysis of primary characteristics that would indicate otherwise. However, experience indicates that this will not be the case. The rate of approach to a uniform state of 100°C depends upon \( \kappa \) as well as the temperature profile in the rod when the end conditions are imposed. Apparently, the single characteristic family defined by equation (1.3.46) is inadequate for fully describing the behavior of the solution.

There is an additional curve that can be obtained from the secondary characteristic analysis, as discussed in the preceding section and presented in Table 1.2. For conduction in a rod with \( D = 1 \) and \( E = 0 \) in the standard equation, the secondary characteristic family has the form
Sec. 1.3 Characteristics and Boundary Conditions

\[ x - x_0 = 0 \]  

(1.3.47)

These characteristics are presented as dashed lines in Figure 1.8 along with the primary characteristics.

For point \( P \) in Figure 1.8, the domain of dependence is the region \( 0 < x < L \) below \( P \), the shaded region. Information propagates toward \( P \) from both ends of the rod along a horizontal characteristic. Because the characteristic is horizontal, information from both ends of the rod is predicted to propagate instantaneously to any point on the interior of the rod (although the instantaneous effect at points far from the boundary would be small). However, the temperature at a point, such as \( P \), is also influenced by the previous temperature profiles in the rod. Thus the secondary characteristics, which emanate from \( t = 0 \), are also significant in calculation of the temperature.

Because only one type of characteristic emanates from the \( x \)-axis into time, only one initial condition must be specified. This condition must involve derivatives of order less than the highest-order time derivative appearing in the differential equation. Thus the appropriate initial condition is specification of \( T \) all along the rod. The primary characteristics are horizontal. Therefore, information may propagate into the rod from both ends, and boundary conditions must be specified at both ends. Because \( T \) is specified as an initial condition, the boundary conditions may be either first type, second type, or third type. The sufficient boundary conditions are presented in Figure 1.9. The parabolic problem studied here is a mixed initial (in time) and boundary (in space) value problem.

The third equation to be considered is the wave equation, which describes, for example, propagation of a disturbance in both directions along a plucked string. The governing equation is

\[ h_{tt} - c^2 h_{xx} = 0 \]  

(1.3.48)

where

- \( h \) is the magnitude of the disturbance,
- \( c \) is the velocity or celerity of the disturbance along the string,
- \( t \) is time, and
- \( x \) is position along the string.
Figure 1.9 Boundary and initial conditions for parabolic equation (1.3.42) solved in region $0 < x < L$ and $t > 0$.

Let the portion of the temporal domain of interest be $0 < t \leq T$ and the portion of the spatial domain be $0 < x < L$. Here the disturbance is assumed small and undamped and the celerity is considered to be constant. With reference to the standard second-order PDE in two independent variables, equation (1.3.16), with $\alpha$ substituted for $u$, $A = -c^2$, $B = 0$, and $C = 1$. Therefore,

$$B^2 - AC = c^2 > 0$$  \hspace{1cm} (1.3.49)

and equation (1.3.48) is hyperbolic with two real characteristics. From equation (1.3.35a) or the third entry in Table 1.1, the equations for the two characteristics are

$$\varphi_{\pm} = \frac{1}{c}$$  \hspace{1cm} (1.3.50a)

$$\varphi_{\pm} = -\frac{1}{c}$$  \hspace{1cm} (1.3.50b)

The solutions to these equations are, respectively,

$$\varphi = \varphi \left( (x - x_0) + c(t - t^0) \right)$$  \hspace{1cm} (1.3.51a)

$$\varphi = \varphi \left( (x - x_0) - c(t - t^0) \right)$$  \hspace{1cm} (1.3.51b)

The actual form of the functional dependence of $\varphi$ on its argument in these equations is arbitrary. However, for a constant value of $\varphi$, the argument will be a constant. For convenience, for some constant $\varphi$, set

$$(x - x_0) + c(t - t^0) = 0$$  \hspace{1cm} (1.3.52a)

$$(x - x_0) - c(t - t^0) = 0$$  \hspace{1cm} (1.3.52b)

Selection of various values for $x_0$ and $t^0$ leads to the plot of characteristics in Figure 1.10. Note that this depicts only representative characteristics for particular values of $x_0$ and $t^0$.

Now that the characteristics have been determined, the question remains as to what are the required boundary conditions to solve equation (1.3.48) in the domain $0 < t \leq T$ and $0 < x < L$. As a first case, consider point $P$ in Figure 1.11, which
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Figure 1.10 Characteristics for hyperbolic equation (1.3.48) in region $0 < x < L$ and $t > 0$ as obtained from equations (1.3.52).

Figure 1.11 Region of influence of the initial conditions for equation (1.3.48) which is unaffected by the boundary conditions at $x = 0$ and $x = L$.

lies at the intersection of the two characteristics that intersect the points $(x,t) = (0,0)$ and $(x,t) = (L,0)$, respectively. Point $P$ has coordinates $(x,t) = (L/2, L/2c)$. The solution at point $P$ will depend on the initial condition, the condition along the $x$-axis at $t = 0$. However, no information specified along the boundary at $x = 0$ or $x = L$ will be transmitted to the point $x = L/2$ until $t > L/2c$. Therefore, to obtain a solution at point $P$, or anywhere in the shaded portion of Figure 1.11, it is only necessary to have specified initial conditions. Because two families of characteristics emanate from the $x$-axis, two conditions must be specified at $t = 0$ for all $x$, these conditions being values of both $h$ and $h_1$ because the order of any derivative in the initial condition for this problem must be less than 2. Specification of both a function and its normal derivative along a boundary is known as a Cauchy boundary condition.

If one shifts point $P$ in the $x$-direction, characteristics that emanate from the boundaries at $x = 0$ and $x = L$ become important, as indicated in Figure 1.12a and b. The location where boundary conditions must be specified, as well as the domain of dependence for the solution at $P$, are indicated, respectively, by wavy lines and shading. Only one characteristic family originates from the $x = 0$ (and $x = L$) boundary. Information propagates forward in time along these characteristics. Therefore, it is necessary to specify only one boundary condition along $x = 0$ and one along $x = L$. Note that for any point $P$ in the region of interest (e.g., Figure 1.12c) the domain of influence is bounded, at least in part, by the $x$-axis along which $h$ and $h_1$ are specified. Because $h$ is
Figure 1.12 Regions of influence of conditions specified on three different portions of the $x$ and $t$ axes (shown by a wavy line) when solving equation (1.3.48).

specified along this part of the boundary, the boundary condition along $x = 0$ or $x = L$ need not include $h$. Sufficient boundary conditions at these positions are specification of either $h$, $h_x$, or a linear combination of the two, such as $\alpha h + \beta h_x$. Figure 1.13 depicts the boundary conditions required to solve hyperbolic equation (1.3.48) in the domain $0 < t \leq T$ and $0 < x < L$.

The differential equation considered is an initial value problem in time because conditions are specified at only the beginning or initial time. However, the equation is a boundary value problem in the spatial domain because conditions must be specified at both ends of the string. Thus the problem of a vibrating string is a mixed initial and boundary value problem.

It is worth noting that the plot of characteristics for the parabolic problem, Figure 1.7, represents a limiting case of the hyperbolic characteristics presented in Figure 1.10. As the magnitude of $c$, the velocity at which a disturbance propagates into
Sec. 1.3 Characteristics and Boundary Conditions

![Graphical representation of boundary and initial conditions for hyperbolic equation](image)

Figure 1.13 Boundary and initial conditions for hyperbolic equation (1.3.48) solved in the region $0 < x < L$ and $t > 0$.

In the domain, becomes infinite, Figure 1.10 becomes identical to Figure 1.7. Thus, from a study of the primary characteristics, the parabolic problem may be thought of as the limit of the hyperbolic problem where the disturbance propagation velocity becomes infinite. Note, however, that for the parabolic problem, secondary characteristics are needed to provide a full description of information propagation.

A final example of a differential equation in two independent variables is the equation describing propagation of a surface elevation disturbance into a channel where flow is uniform (i.e., flow velocity and depth are initially constant in the channel). The governing equation is

$$h_{tt} + 2vh_{tx} + (v^2 - c^2)h_{xx} = 0$$

(1.3.53)

where

- $v$ is the positive constant uniform flow velocity,
- $c$ is the magnitude of the velocity of the disturbance, or celerity,
- $h$ is the amplitude of the surface disturbance,
- $t$ is time, and
- $x$ is space.

Let the temporal domain be $0 < t < T$ and the spatial domain be $0 < x < L$. Note that when the flow velocity is zero, this equation becomes identical to equation (1.3.48), the equation for propagation of a disturbance along a string.

With reference to the standard PDE of equation (1.3.16), $A = v^2 - c^2$, $B = v$, $C = 1$, and $D = E = R = 0$. Therefore,

$$B^2 - AC = c^2 > 0$$

(1.3.54)

and equation (1.3.53) is hyperbolic with two real characteristics. The equations for the characteristics are, from Table 1.1,

$$\varphi_t = (c - v)\varphi_x$$

(1.3.55a)

and

$$\varphi_t = (c + v)\varphi_x$$

(1.3.55b)
Thus the characteristics are
\[ \varphi = \varphi \left( (x - x_0) + (c - v)(t - t^0) \right) \]  
(1.3.56a)
and
\[ \varphi = \varphi \left( (x - x_0) - (c + v)(t - t^0) \right) \]  
(1.3.56b)
or
\[ (x - x_0) + (c - v)(t - t^0) = 0 \]  
(1.3.57a)
\[ (x - x_0) - (c + v)(t - t^0) = 0 \]  
(1.3.57b)

For these characteristics, three regimes can be identified that require different boundary conditions. The first regime is for the case where \( c > v \). The characteristics for this situation are sketched in Figure 1.14a. Note that two families emanate from the \( x \)-axis while one family emanates from each end of the domain. Figure 1.14b shows the domain of dependence for point \( P \). Solutions in this domain depend only on the conditions specified at \( t = 0 \), not upon conditions set at \( x = 0 \) or \( x = L \). Because two families of characteristics intersect the \( x \)-axis, two initial conditions are required, both \( h \) and \( h_x \). Note that conditions specified at \( x = 0 \) and \( x = L \) propagate along characteristics in the direction of increasing time. For the case of \( c > v \), this information propagates into the domain of interest. Therefore, to obtain a solution for the total domain of the problem, one additional condition must be specified at \( x = 0 \) and one at \( x = L \). This condition may be either \( h \), \( h_x \), or a linear combination of the two. The boundary conditions needed to obtain a solution to equation (1.3.53) where \( c > v \) in the domain \( 0 < t \leq T \) and \( 0 < x < L \) are indicated in Figure 1.14c. This problem is an initial value problem in time and a boundary value problem in space.

The second regime of interest in solving equation (1.3.53) is when \( c < v \). From equations (1.3.57) it can be seen that both characteristics have a positive slope when plotted in the \( x-t \) plane as in Figure 1.15a. In this case, point \( P \), the point of maximum time that is uninfluenced by conditions imposed at the ends of the domain, is located at \( x = L \). The domain of dependence for the solution at \( P \) is indicated in Figure 1.15b. Note that the solution in this region is dependent only on two specified initial conditions, \( h \) and \( h_x \), because information specified at \( x = L \) does not propagate into the system but propagates forward in time to \( x \) locations greater than \( L \). Physically, the velocity of the flow is greater than the upstream celerity of a condition imposed at \( x = L \), and thus any condition imposed there is washed out of the system. Figure 1.15c indicates that the solution to equation (1.3.53) above the domain of dependence of point \( P \) for \( c < v \) is independent of initial conditions but depends on information propagating along two characteristic families from the \( x = 0 \) axis. Thus both \( h \) and \( h_x \) must be specified at \( x = 0 \) and no conditions are specified at \( x = L \). Solution of equation (1.3.53) in the entire domain of interest when \( c < v \) requires conditions as indicated in Figure 1.15d. Thus when the velocity of the flow is greater than the celerity of the disturbance, the differential equation is an initial value problem in both time and space.
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(1.3.55b)

(1.3.56a)

(1.3.56b)

(1.3.57a)

(1.3.57b)

Figure 1.14 For hyperbolic equation (1.3.53) with $c > v$ and characteristics given by equations (1.3.57): (a) Characteristics in region $0 < x < L$ and $t > 0$. (b) Region of influence of the initial conditions which is unaffected by the boundary condition at $x = 0$ and $x = L$. (c) Boundary and initial conditions in region $0 < x < L$ and $t > 0$.

The third possible regime for equation (1.3.53) is the case where $c = v$. Characteristics for this situation are plotted in Figure 1.16a. In this regime, as with the second regime, information specified at the downstream boundary, $x = L$, does not propagate back into the problem domain. Note that two families of characteristics intersect the $x$-axis while only one family intersects the $t$-axis. The domain where the solution is
Figure 1.15 For hyperbolic equation (1.3.53) with $c < v$ and characteristics given by equation (1.3.57): (a) Characteristics in region $0 < x < L$ and $t > 0$. (b) Region of influence of the initial conditions where boundary conditions at $x = 0$ have no effect. (c) Region of influence of the boundary conditions at $x = 0$ where the initial conditions have no effect.

dependent only on the initial conditions and is independent of conditions imposed at $x = 0$ and $L$ is indicated in Figure 1.16b. Because two characteristics intersect the $x$-axis, both $h$ and $h_x$ must be specified at the initial time. The characteristic family that intersects the time axis at $x = 0$ affects the flow in the problem domain above the point
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Figure 1.15 (cont'd.) For hyperbolic equation (1.3.53) with \( c < v \) and characteristics given by equation (1.3.57): (d) Boundary and initial conditions in region \( 0 < x < L \) and \( t > 0 \).

\( P \) region of influence. Thus only one condition must be specified at \( x = 0 \) and none are specified at \( x = L \), as indicated in Figure 1.16c. When \( v = c \), equation (1.3.53) is an initial value problem in both time and space.

This last example demonstrates that for a hyperbolic problem, knowledge of the characteristics is essential for imposition of appropriate boundary conditions. Depending on the value of parameters in the problem, the locations where boundary conditions must be imposed and even the number of conditions needed to obtain the solution to a differential equation can be affected.

The examples above are simple in that the coefficients of the equations were selected to be constant. When these coefficients are not constant, the equation type can be different in different regions of the solution domain and the characteristic lines can be curves. Furthermore, explicit solutions for the characteristics may not be readily obtained. However, determination of equation type for the second-order problem in two-independent variables, equation (1.3.16), provides information about required boundary conditions for many problems of physical interest. Elliptic equations are boundary value problems, independent of time, and must be solved in a closed region. A boundary condition is required at each point on the boundary. A parabolic equation is commonly (though not always) a transient problem. The equation is an initial value problem in time and a boundary value problem in space. The region of solution is open-ended in time (i.e., extends to infinity), and an initial condition in time is required, together with boundary conditions at both ends of the spatial domain. A hyperbolic equation requires specification of two initial conditions. Typically, two auxiliary conditions are required in the spatial domain. However, whether the problem is an initial value or boundary value problem in space depends on the celerity, the speed of propagation of a disturbance.

1.3.4 Second-Order Partial Differential Equations (Three Independent Variables)

In Section 1.3.2 second-order partial differential equations were classified as elliptic, parabolic, or hyperbolic, depending on whether the equation had zero, one, or two real
characteristic curves, respectively. This classification scheme can be extended to the case of three independent variables except that here the characteristic curves are surfaces, and not all equations fall into one of these three categories.
Consider the quasilinear second-order PDE
\[ Au_{tt} + Bu_{xx} + Cu_{yy} + 2D u_{xy} + 2Eu_{xt} + Gu_t + Hu_x + Ku_y = R \] (1.3.58)
where \( A, B, C, D, E, F, G, H, \) and \( K \) may be functions of \( x, y, t, u_x, u_y, u_t, \) and \( u, \) and \( R \) may depend on \( x, y, t, \) and \( u. \) This equation may be converted to a first-order form by defining
\[ f = u_t \] (1.3.59)
\[ g = u_x \] (1.3.60)
\[ h = u_y \] (1.3.61)
Then note that
\[ g_y = h_x \] (1.3.62)
\[ f_x = g_t \] (1.3.63)
\[ f_y = h_t \] (1.3.64)
and replace second derivatives of \( u \) in equation (1.3.58) with first derivatives of \( f, g, \) and \( h, \) to obtain
\[ Af_x + Bg_x + Ch_y + D(g_y + h_x) + E(f_x + g_t) + F(f_y + h_t) + G u_t + H u_x + K u_y = R \] (1.3.65)
These last seven equations form the basis of the following effort to define surfaces on which specification of \( u, f, g, \) and \( h \) is insufficient information for determination of \( u \) throughout a domain of interest. These surfaces are the characteristics of the equation.
For problems in three independent variables, surfaces in \( x, y, \) and \( t, \) or \( x, t, \) space are examined. These surfaces may be defined as
\[ \varphi(x, y, t) = \varphi(x, t) = \text{constant} \] (1.3.66)
If one moves from point \((x, t)\) to point \((x + dx, t + dt)\) the function \( \varphi \) will undergo a change in value \( d\varphi \) where
\[ d\varphi = \varphi_x dx + \varphi_y dy + \varphi_t dt \] (1.3.67)
If the movement is constrained such that \( \varphi \) remains constant, then
\[ D \varphi = 0 = \varphi_x dx + \varphi_y dy + \varphi_t dt \] (1.3.68)
where the notational change from \( d \) to \( D \) is used to indicate movement in space-time on the surface defined by \( \varphi = \text{constant}. \) If the motion is further constrained to take place with \( y \) constant, then \( D_y = 0 \) and equation (1.3.68) becomes
\[ \left( \frac{Dt}{Dx} \right)_y = -\frac{\varphi_x}{\varphi_t} \] (1.3.69)
where the \( y \) subscripted on \( Dt/Dx \) is used to indicate that \( y \) is held constant. Similarly, if \( Dx \) is zero, motion on \( \varphi \) occurs with only \( Dy \) and \( Dt \) changing and
\[ \left( \frac{Dt}{Dy} \right)_x = -\frac{\varphi_y}{\varphi_t} \] (1.3.70)
Now consider the change in $u$ due to movement from $(x, t)$ to $(x + dx, t + dt)$. This infinitesimal change is denoted by

$$du = u_x dx + u_y dy + u_t dt$$  \hspace{1cm} (1.3.71)

If the motion occurs on a surface $\varphi$, this is indicated by

$$Du = u_x D\varphi_x + u_y D\varphi_y + u_t D\varphi_t$$  \hspace{1cm} (1.3.72)

where $Du$ is the change in $u$ obtained due to movement on the surface of constant $\varphi$. If the movement is constrained such that $D\varphi_y = 0$ as well as $D\varphi_t$, then, after division by $D\varphi$, equation (1.3.72) becomes

$$u_x = -u_t \left( \frac{Dt}{D\varphi} \right)_y + \left( \frac{Du}{D\varphi} \right)_y$$  \hspace{1cm} (1.3.73)

However, equation (1.3.69) provides an alternative expression for $(Dt/D\varphi)_y$ that can be substituted into equation (1.3.73) to obtain

$$u_x = u_t \frac{\varphi_x}{\varphi_t} + \left( \frac{Du}{D\varphi} \right)_y$$  \hspace{1cm} (1.3.74a)

By an analogous series of steps in which $x$ and $y$ are interchanged in the derivation following equation (1.3.72),

$$u_y = u_t \frac{\varphi_y}{\varphi_t} + \left( \frac{Du}{D\varphi} \right)_x$$  \hspace{1cm} (1.3.74b)

Manipulations identical to those performed on $u$ in this paragraph can be performed on $f$, $g$, and $h$ to obtain

$$f_x = f_t \frac{\varphi_x}{\varphi_t} + \left( \frac{Df}{D\varphi} \right)_y$$  \hspace{1cm} (1.3.75a)

$$f_y = f_t \frac{\varphi_y}{\varphi_t} + \left( \frac{Df}{D\varphi} \right)_x$$  \hspace{1cm} (1.3.75b)

$$g_x = g_t \frac{\varphi_x}{\varphi_t} + \left( \frac{Dg}{D\varphi} \right)_y$$  \hspace{1cm} (1.3.76a)

$$g_y = g_t \frac{\varphi_y}{\varphi_t} + \left( \frac{Dg}{D\varphi} \right)_x$$  \hspace{1cm} (1.3.76b)

$$h_x = h_t \frac{\varphi_x}{\varphi_t} + \left( \frac{Dh}{D\varphi} \right)_y$$  \hspace{1cm} (1.3.77a)

$$h_y = h_t \frac{\varphi_y}{\varphi_t} + \left( \frac{Dh}{D\varphi} \right)_x$$  \hspace{1cm} (1.3.77b)

If $u$, $f$, $g$, and $h$ are specified on a surface $\varphi$, these last eight equations provide expressions for partial derivatives of these functions with respect to a spatial variable in terms of a partial derivative with respect to time and some known quantities. [For example, if $h$ is known on a specified surface $\varphi$, then $\varphi_x$, $\varphi_y$, $\varphi_t$, $( Dh/ Dy )_x$, and $( Dh/ D\varphi )_y$ can
be calculated so that \( h_x \) and \( h_y \) in equation (1.3.77) may be expressed in terms of \( h_t \) and these known quantities.]

Substitution of equations (1.3.74)–(1.3.77) into (1.3.65) to eliminate derivatives with respect to spatial coordinates followed by regrouping of terms yields

\[
\frac{u_t (G \varphi_t + H \varphi_x + K \varphi_y)}{\varphi_t} + \frac{f_t (A \varphi_t + E \varphi_x + F \varphi_y)}{\varphi_t} + \frac{g_t (E \varphi_t + B \varphi_x + D \varphi_y)}{\varphi_t} + \frac{h_t (F \varphi_t + D \varphi_x + C \varphi_y)}{\varphi_t} = \mathcal{R} \tag{1.3.78}
\]

where the known quantities are grouped in \( \mathcal{R} \), which is given by

\[
\mathcal{R} = R - B \left( \frac{Du}{Dx} \right)_y - C \left( \frac{Dh}{Dy} \right)_x - D \left( \frac{Dg}{Dy} \right)_x - D \left( \frac{Dh}{Dx} \right)_y - E \left( \frac{Df}{Dx} \right)_y - F \left( \frac{Df}{Dy} \right)_x - H \left( \frac{Du}{Dy} \right)_x - K \left( \frac{Du}{Dy} \right)_x \tag{1.3.79}
\]

Now recall equation (1.3.59):

\[
u_t = f \tag{1.3.80}
\]

Also combine (1.3.63) with (1.3.75a) to obtain

\[
g_t = f_t + \frac{\varphi_x}{\varphi_t} \left( \frac{Df}{Dx} \right)_y \tag{1.3.81}
\]

and combine (1.3.64) with (1.3.75b) to obtain

\[
h_t = f_t + \frac{\varphi_y}{\varphi_t} \left( \frac{Df}{Dy} \right)_x \tag{1.3.82}
\]

These last three equations can be combined with equation (1.3.78) to obtain the matrix equation

\[
\begin{bmatrix}
G \varphi_t + H \varphi_x + K \varphi_y \\
A \varphi_t + E \varphi_x + F \varphi_y \\
E \varphi_t + B \varphi_x + D \varphi_y \\
F \varphi_t + D \varphi_x + C \varphi_y
\end{bmatrix}
\begin{bmatrix}
\varphi_t \\
\varphi_t \\
\varphi_t \\
\varphi_t
\end{bmatrix}
= \begin{bmatrix}
\mathcal{R} \\
f_t \\
g_t \\
h_t
\end{bmatrix} \tag{1.3.83}
\]

This equation indicates that \( u_t, f_t, g_t, \) and \( h_t \) may be obtained on a surface \( \varphi \) where \( u, f, g, \) and \( h \) are specified unless the determinant of the matrix is zero. If the determinant
is zero, the surface \( \varphi \) is a characteristic. Thus the equation of the characteristic may be obtained by setting the determinant equal to zero, or

\[
A \varphi_x^2 + B \varphi_y^2 + C \varphi_z^2 + 2D \varphi_x \varphi_y + 2E \varphi_x \varphi_z + 2F \varphi_y \varphi_z = 0
\]  
(1.3.84)

The equation for the secondary characteristic (which is the primary characteristic of the first-order equation obtained when \( A, B, C, D, E, \) and \( F \) are all zero) is given by

\[
G \varphi_x + H \varphi_y + K \varphi_z = 0
\]  
(1.3.85)

The explicit determination of characteristics for a second-order differential equation in three independent variables is, in general, complex. A table such as Table 1.1 prepared for characteristics of equations in two independent variables is not readily constructed. In the next section, some relatively simple though important specific equations in three variables are analyzed.

### 1.3.5 Example Second-Order Partial Differential Equations

(Three Independent Variables)

For steady-state conduction in a homogeneous isotropic block of material, the governing equation is

\[
T_{xx} + T_{yy} + T_{zz} = 0
\]  
(1.3.86)

If \( z \) is used in place of \( t \) in the model equation (1.3.58), then \( A = B = C = 1 \) and the other coefficients are zero. The equation for the characteristics, (1.3.84), becomes

\[
\varphi_x^2 + \varphi_y^2 + \varphi_z^2 = 0
\]  
(1.3.87)

This equation cannot be satisfied if \( \varphi_x, \varphi_y, \text{ and } \varphi_z \) are constrained to be real with at least one of these terms being nonzero. Therefore, no real characteristics exist and equation (1.3.86) is elliptic. Either \( T, T_n \) (the normal gradient of \( T \)), or a linear combination of these quantities must be specified at every point on the boundary surface. As with the two-independent-variable case, \( T \) must be specified for at least one point on the boundary to ensure a unique solution.

A second example that indicates some of the utility of the characteristic analysis is the equation describing transient transport of a dissolved chemical species. Assume that this process is to be modeled as pure advection in the \( x \)-direction and pure diffusion in the \( y \)-direction. Let the spatial boundaries of the domain be \( 0 < x < L \) and \( 0 < y < b \). The equation describing this process is

\[
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial y^2} = 0
\]  
(1.3.88)

where the velocity, \( v \), and diffusion coefficient, \( D \), are constants. This equation corresponds to equation (1.3.53) with \( G = 1, H = v \), and \( C = -D \). From equation (1.3.84), the characteristic equation is

\[
-D \varphi_y^2 = 0
\]  
(1.3.89)
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This expression has a double root and describes a characteristic, \( \varphi(x - x_0, t - t^0) \), which spans the \( y \)-domain of the problem for any fixed coordinate values of \( x \) and \( t \). Thus boundary conditions are needed at both \( y \) boundaries. Equation (1.3.89) defines only one unique characteristic family, but another may be obtained from the secondary characteristic equation (1.3.85) as

\[
\varphi_t + v\varphi_x = 0 \tag{1.3.90}
\]

Therefore, the secondary characteristic has the functional form

\[
\varphi = \varphi \left( (x - x_0) - v(t - t^0), y - y_0 \right) \tag{1.3.91}
\]

This secondary characteristic, at a fixed \( y \) location, demonstrates that information propagates into the system from the \( t = 0 \) axis and from the \( x = 0 \) position. Thus auxiliary conditions are needed at the initial time \( (t = 0) \) and at the inlet \( (x = 0) \). Notice that no information need be prescribed at the outflow boundary in the \( x \)-direction. Equation (1.3.88) may be thought of as being hyperbolic with respect to \( t \) and \( x \) and parabolic with respect to \( t \) and \( y \).

It is very important to note that if the \( x \)-direction velocity, \( v \), and the \( y \)-direction width, \( b \), were allowed to be functions of \( x \), with the constraint that \( vb = \) constant still imposed, the application of boundary conditions for equation (1.3.88) is greatly complicated. (This case is discussed in Problems 1.5 and 1.6.) Often the reason that a numerical solution to a problem is sought rather than an analytic solution is that the boundaries of the domain of interest are not parallel to coordinate axes. In these instances, application of appropriate boundary conditions can be a matter of art and requires engineering judgment as well as mathematical rigor.

As a third example, consider the equation of propagation of a disturbance in a two-dimensional domain, such as the movement of sound waves in a drumhead. Dissipation of the wave will be neglected. The governing differential equation is

\[
h_{tt} - c^2(h_{xx} + h_{yy}) = 0 \tag{1.3.92}
\]

where \( c \) is the wave celerity and \( h \) is the displacement of a point on the drumhead from its rest position. Based on the general equation (1.3.58), \( A = 1, B = -c^2, C = -c^2 \), and the other coefficients are zero. Then the expression for the characteristics can be obtained from equation (1.3.84) as

\[
\varphi_t^2 - c^2(\varphi_x^2 + \varphi_y^2) = 0 \tag{1.3.93}
\]

which has the solutions

\[
\varphi = \varphi \left( c(t - t^0) + [(x - x_0)^2 + (y - y_0)^2]^{1/2} \right) \tag{1.3.94a}
\]

and

\[
\varphi = \varphi \left( c(t - t^0) - [(x - x_0)^2 + (y - y_0)^2]^{1/2} \right) \tag{1.3.94b}
\]

Therefore, the equations for the characteristic surfaces are the conoidal sections

\[
c(t - t^0) + [(x - x_0)^2 + (y - y_0)^2]^{1/2} = 0 \tag{1.3.95a}
\]
and
\[ c(t - t^0) - \sqrt{(x - x_0)^2 + (y - y_0)^2} = 0 \] (1.3.95b)

Because two real characteristics exist, equation (1.3.92) is hyperbolic. As with the example in one space dimension, because two characteristics emanate from the constant time plane, two initial conditions are required. In addition, one condition is needed at each point on the spatial boundary. For the case of the drumhead, the initial conditions would be the state of displacement and the rate of change of this displacement \((h_t \text{ and } h_x)\) while the boundary conditions would be zero displacement \((h = 0)\) all along the rim of the drum.

These three examples provide some insight into the use of characteristics to obtain appropriate boundary conditions for multidimensional problems. However, not all equations can be analyzed easily using characteristics. For example, when the coefficients in equation (1.3.58) are strong functions of space, time, or the first derivatives, the analysis to obtain characteristics may become virtually impossible. Also, the geometry of the characteristic surfaces can be complex. In these instances, specification of the correct boundary conditions for a numerical simulation is subject to approximations. Irregular geometry of the solution domain can also contribute to difficulty in specifying appropriate boundary conditions. Finally, the analysis here has been restricted to quasilinear first- and second-order differential equations. For nonlinear equations the analysis for characteristics is more complex. However, virtually all equations of physical interest can be classified.

1.3.6 Second-Order Partial Differential Equations (Four Independent Variables)

The derivations in the preceding section for the characteristics of second-order differential equations may be extended to equations in four or more independent variables. For these cases, the characteristics are hypersurfaces. The derivation follows the same reasoning as was used in the previous cases and is left as an exercise in Problem 1.7.

Without performing the formal derivation of the equation for the characteristics, the result can be inferred by comparison of equations (1.3.84) and (1.3.85) with equation (1.3.58). The primary characteristics depend only on the coefficients of the second derivatives. These coefficients multiply products of first derivatives of the characteristic functions in the corresponding independent variables. The secondary characteristics are obtained in a similar manner using the first derivative terms.

For example, the three-dimensional transient equation describing advection and diffusion of a reacting chemical species is
\[ u_t + U u_x + V u_y + W u_z - D(u_{xx} + u_{yy} + u_{zz}) - k u = 0 \] (1.3.96)

where \(u\) is concentration; \(U, V,\) and \(W\) are \(x, y,\) and \(z\) velocity components, respectively, \(D\) is the diffusion coefficient, and \(k\) is the reaction rate constant. This equation has characteristics given by
\[ \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \] (1.3.97)
and secondary characteristics given by
\[ \varphi_t + U\varphi_x + V\varphi_y + W\varphi_z = 0 \]  
(1.3.98)

From equation (1.3.97), only one real characteristic may be obtained,
\[ \varphi = \varphi(t - \delta^0) \]  
(1.3.99a)

or
\[ t - \delta^0 = 0 \]  
(1.3.99b)

Thus equation (1.3.96) is parabolic and the secondary characteristic is of interest. If \( U \), \( V \), and \( W \) are constant, solution of equation (1.3.98) yields
\[ \varphi \left( (U^2 + V^2 + W^2)(t - \delta^0) - U(x - x_0) - V(y - y_0) - W(z - z_0) \right) = 0 \]  
(1.3.100a)

or
\[ (U^2 + V^2 + W^2)(t - \delta^0) - U(x - x_0) - V(y - y_0) - W(z - z_0) = 0 \]  
(1.3.100b)

If \( U \), \( V \), or \( W \) is not constant, the secondary characteristics will include some curvature. Otherwise, equation (1.3.100b) defines a hyperplane that is analogous to the simpler curves of equation (1.3.15b).

Numerical simulation in time and three spatial dimensions becomes very difficult because large amounts of computer time and storage are typically needed, irregular geometry of a domain of interest is difficult to incorporate into a model, differential equations being simulated are often quasilinear or nonlinear, and knowledge and proper application of boundary conditions is complex. Preliminary analytical analysis for characteristics of such an equation may not yield explicit forms of the characteristics, but it can provide insight into the behavior of the solution and guidance in the selection of an appropriate numerical method.

1.4 CONCLUSION

This chapter is intended to provide some physical as well as mathematical understanding of the behavior of partial differential equations through analysis of their characteristics and boundary conditions. The examples presented are relatively simple in that they involve linear equations with constant coefficients. Therefore, processes such as the formation of shocks or those which cause the classification of a differential equation to be dependent on location in the solution domain are not considered. Additionally, in all the examples presented the characteristics did not exhibit any curvature. Nevertheless, the insights that can be gained from understanding of the mechanisms of information propagation along characteristics are very helpful in developing computational algorithms capable of capturing the physics of a problem.

At times numerical simulation is viewed abstractly as an approximate tool for solution of a differential equation. However, when a simulation is attempted in the context of understanding of the physical processes of interest, the opportunity for a real growth in knowledge is greatly enhanced. Characteristics are valuable tools that
contribute to such understanding, assist in the identification of necessary and sufficient boundary conditions, and help in formulation and selection of computational algorithms.

**PROBLEMS**

1.1. Classify the following differential equations (1) as ordinary or partial; (2) as to order; (3) as to degree; (4) as linear, nonlinear, or quasilinear; and (5) as homogeneous or nonhomogeneous.

(a) \[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left( D \frac{\partial u}{\partial x} \right) + ku^2 = 0 \]
where \( v, D, \) and \( k \) are constants

(b) \[ \frac{\partial T}{\partial t} - \frac{\partial}{\partial x} \left( \kappa \frac{\partial T}{\partial x} \right) = 0 \]
where \( \kappa = \kappa(T) \)

(c) \[ \frac{\partial \zeta}{\partial t} + h \frac{\partial u}{\partial x} = 0 \]
\[ \frac{\partial u}{\partial t} + \tau u + g \frac{\partial \zeta}{\partial x} = 0 \]
where \( h, \tau, \) and \( g \) are constants

(d) \[ \frac{d^2 f}{dx^2} + M \left[ 1 + \left( \frac{df}{dx} \right)^2 \right]^{1.5} = 0 \]
where \( M = M(x) \)

(e) \[ S \frac{\partial h}{\partial t} - \frac{\partial}{\partial x} \left( T \frac{\partial h}{\partial x} \right) - \frac{\partial}{\partial y} \left( T \frac{\partial h}{\partial y} \right) = Q \]
where
\[ Q = Q(x, y, t) \]
\[ T = T(x, y, h) \]
\[ S = S(x, y) \]

1.2. Given the differential equation
\[ \frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = 0 \]
where \( u = u(x, t) \), \( x \) has units of feet, and \( t \) has units of seconds, determine the following solutions, if possible.

(a) If \( u(0, t) = 1, u(x, 0) = 0 \), and \( v = 2 \) ft/sec,
fund \( u(1, 1), u(3, 2), \) and \( u(7, 2) \).

(b) If \( u(0, t) = 1, u \left( x, \frac{1}{2} \right) = 2 \), and \( v = 1 \) ft/sec,
fund \( u(1, 3), u(1, 2), u(4, 1), \) and \( u(4, 4) \).

(c) If \( u(0, t) = f/sec, u(x, 0) = \frac{x}{f}, \) and \( v = 3 \) ft/sec,
fund \( u(1, 1), u(3, 2), u(4, 4), \) and \( u(1, 3) \).

(d) If \( u(0, t) = f/sec, u(x, 0) = \frac{x}{f}, \) and \( v = \frac{x}{sec}, \)
fund \( u(1, 1), u(3, 2), u(4, 4), \) and \( u(1, 3) \).

(e) If \( u(0, t) = 1 \) for \( 3 \geq t > 0 \) and \( u(x, x/2 - 1) = 2 \), and \( v = 1 \) ft/sec, find the domain
where a solution may be obtained and the solution within that domain.
Problems

1.3. Equation (1.3.16), the second-order PDE in $x$ and $t$, can be reexpressed by equations (1.3.17) through (1.3.20). Develop a series of manipulations analogous to those in Section 1.3.2 to obtain the equation for characteristics as follows.

(a) Show that

$$u_t = \frac{Du}{Dt} + u_x \frac{\varphi_t}{\varphi_x}$$

$$f_t = \frac{Df}{Dt} + f_x \frac{\varphi_t}{\varphi_x}$$

$$g_t = \frac{Dg}{Dt} + g_x \frac{\varphi_t}{\varphi_x}$$

(b) Use the relations in part (a) to eliminate $u_t$, $f_t$, and $g_t$ from equation (1.3.19).

(c) Use equation (1.3.18), $g = u_x$, and equation (1.3.27c), with $g_t$ replaced by $f_x$, together with the result of part (b) to obtain a $3 \times 3$ matrix equation with $u_x$, $f_x$, and $g_x$ as unknowns.

(d) Obtain the equation for the characteristics and compare with equation (1.3.34).

1.4. The equation describing two-dimensional, steady-state diffusion and first-order decay of a chemical species is

$$\mathcal{D}[u_{xx} + u_{yy}] + ku = 0$$

where $\mathcal{D}$ is the constant diffusion coefficient and $k$ is the chemical reaction rate constant. Assume that this equation applies to a two-dimensional region $\Omega$ with boundary $\partial \Omega$.

(a) Classify this equation as elliptic, hyperbolic, or parabolic and obtain the equation for the characteristics.

(b) Let the boundary conditions for this problem be of Neumann type such that

$$\frac{\partial u}{\partial n} = f(x, y) \text{ on } \partial \Omega$$

where $n$ is the direction normal to the boundary. If $u_{x1}$ is a solution to this problem such that it satisfies the differential equation and the boundary conditions, show that this solution is unique unless $k$ or $f$ is zero.

1.5. Consider flow and transport between two plates of length $L$ where the distance between the plates is given by $b = b_0 + \alpha x$, as shown in the figure. Assume that all properties are constant in the direction normal to $x$ and $y$. If diffusion in the $x$-direction is considered negligible, the velocity in the $x$-direction is independent of $y$, and the flow rate through the aperture is constant, then the equation describing the transport is

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} - \mathcal{D} \frac{\partial^2 u}{\partial y^2} = 0$$

where $\mathcal{D} = Q = \text{ constant}$.

(a) Show that the characteristics for this problem are given by $\varphi_y = 0$ such that

$$\varphi(x - x_0, t - t^0) = 0$$

(b) Show that the secondary characteristic has the functional form

$$\varphi = \varphi \left\{ \frac{1}{Q} \left[ b_0(x - x_0) + \frac{1}{2} \alpha x^2 \right] - (t - t^0), y - y_0 \right\}$$
(c) Sketch the primary and secondary characteristics for this problem.
(d) The following boundary conditions have been proposed for this problem for the function $u(x, y, t)$:
   (i) $u(0, y, t) = f(y)$ where $f(y)$ is known
   (ii) $\frac{\partial u}{\partial n} \left( x, \frac{1}{2}(\alpha x + b_0), t \right) = 0$
   (iii) $\frac{\partial u}{\partial n} \left( x, -\frac{1}{2}(\alpha x + b_0), t \right) = 0$
   (iv) $u(x, y, 0) = 0$
Note that conditions (ii) and (iii) indicate no diffusion into the walls of the aperture. Discuss whether or not these conditions are sufficient to obtain a unique solution to the problem.

1.6. One proposal for simplifying the analysis and solution of Problem 1.5 is to include a term in the governing equation that allows for a very small amount of diffusion in the $x$-direction. Thus the problem solution is essentially unchanged, but the equation is modified to the form

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial y^2} - \epsilon \frac{\partial^2 u}{\partial x^2} = 0$$

where $\epsilon$ is very small.

(a) Obtain the characteristics for this equation and classify the equation by type.
(b) Obtain the secondary characteristics, if any.
(c) Sketch the characteristics for the problem.
(d) Pose suitable boundary conditions for the problem.
(e) Discuss differences between this problem and Problem 1.5.

1.7. (a) Write the quasilinear second-order PDE analogous to equation (1.3,58) with three spatial variables and time.

(b) Show that if the characteristics are defined by $\varphi(x, t) = \text{constant}$, then

$$\left( \frac{Dt}{Dx} \right)_{y,z} = -\frac{\varphi_x}{\varphi_t}$$
$$\left( \frac{Dt}{Dy} \right)_{x,z} = -\frac{\varphi_y}{\varphi_t}$$
$$\left( \frac{Dt}{Dz} \right)_{x,y} = -\frac{\varphi_z}{\varphi_t}$$
(c) Show that for a function $u$,

$$u_x = u_t \frac{\varphi_x}{\varphi_t} + \left( \frac{Du}{Dx} \right)_{y,z}$$

$$u_y = u_t \frac{\varphi_y}{\varphi_t} + \left( \frac{Du}{Dy} \right)_{x,z}$$

(d) Obtain the equations of the characteristics and the secondary characteristics.