Aside: The Total Differential
Motivation: Where does the concept for “exact ODEs” come from?

If a multivariate function $u(x,y)$ has continuous partial derivatives, its differential, called a Total Derivative, is given by

$$ du = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy $$

or, “the change in $u$ is given by the change in $u$ with respect to $x$ plus the change with respect to $y$. ” We use such total differentials throughout hydrology. You’ve seen it most recently in your hydrogeochemistry course, regarding Gibb’s free energy $G$, as $u$, and enthalpy $H$ and entropy $S$, as $x$ and $y$, respectively, or $dG = dH – T \, dS$, where $T$ is absolute temperature.

In any event, it follows that if $u(x,y) = c = \text{const.}$, then $du = 0$. We can use this to suggest a method to solve first order ODEs.

Example (from the text):

Let $u = x + x^2y^3 = \text{some constant, say } b$, then

$$ du = \frac{\partial(x + x^2y^3)}{\partial x} \, dx + \frac{\partial(x + x^2y^3)}{\partial y} \, dy = \left[ \frac{\partial x}{\partial x} + y^3 \frac{\partial x^2}{\partial x} \right] \, dx + \left[ \frac{\partial x}{\partial y} + x^2 \frac{\partial y^3}{\partial y} \right] \, dy $$

$$ = (1 + 2xy^3) \, dx + 3x^2y^2 \, dy = db = 0. $$

Or,

$$ \frac{dy}{dx} = -\frac{1 + 2xy^3}{3x^2y^2}. $$

This is an ODE that we can solve by “going backwards”, since we already know that the solution is $u = x + x^2y^3 = b$.

This suggests the powerful solution method we know as the method for exact first-order ordinary differential equations. In this case we take the ODE and rewrite it as a differential, then test to see if it is exact. If not, we test to see if we can make it exact through an integrating factor. In either case, if we end up with an exact equation we can solve it using this method.

---

3 Here dependent variable $u$ is a function of two independent variables, $x$ and $y$. We’ll review multivariate calculus in detail just before introducing vector calculus and PDEs.
**Exact ODE Example:**

Given the ODE (not in the text)

\[
\frac{dy}{dx} = -\frac{3x^2 + y^2}{2xy} = 0
\]

Step 1. Rewrite as a differential

\[(3x^2 + y^2)dx + 2xy \, dy = 0\]

Step 2. Could we solve this by SOV? No. The argument on the 1st \((dx)\) term is not separable.

Step 3. Then test to see if it is an exact equation. That is, when written as

\[M(x, y) \, dx + N(x, y) \, dy = 0\]

what is \(M\), what is \(N\), and do they satisfy the test for “exactness”, which is

\[\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\]

In this example, \(M = 3x^2 + y^2\) and \(N = 2xy\), so that

\[\frac{\partial M}{\partial y} = 2y \quad \text{and} \quad \frac{\partial N}{\partial x} = 2y,\]

which equal to each other, thus passing the test. Our ODE is exact. If it wasn’t we would look for an integrating factor. If we couldn’t find one that leads to exactness, we would seek another method.

Step 4. Find the solution. There are two routes, one through \(M\) (on the left below) and one through \(N\) (on the right). The first uses \(\partial u/\partial x=M\), while the second uses \(\partial u/\partial y=N\). We’ll examine both for completeness.

**Route through \(M\)**

\[
u = \int M \, dx + k(y) = \int (3x^2 + y^2) \, dx + k(y) = \frac{3x^3}{3} + y^2x + k(y)
\]

**Route through \(N\)**

\[
u = \int N \, dy + l(x) = \int 2xy \, dy + l(x) = xy^2 + l(x)
\]
\[ \frac{\partial u}{\partial y} = \frac{\partial x^3}{\partial y} + x \frac{\partial y^2}{\partial y} + \frac{dk}{dy} \]
\[ = 2xy + \frac{dk}{dy} = N + \frac{dk}{dy} \]
\[ = N \text{, therefore } \frac{dk}{dy} = 0 \]
and \( k = \text{const.} \)
\[ \therefore u = x^3 + y^2x + k \]
But \( u = c = \text{const}, \) therefore
(with \( c^* = \text{const} = c-k \))
\[ x^3 + y^2x = c^* \]
is a solution, where \( c^* \) is a constant

\[ \frac{\partial u}{\partial x} = y^2 + \frac{dl}{dx} \]
\[ = M = 3x^2 + y^2 \]
therefore \( \frac{dl}{dx} = 3x^2 \)
and \( l = x^3 + b \)
where \( b \) is a constant
\[ u = xy^2 + l = y^2x + x^3 + b \]
compare to first column \( k = b \)
\[ u = x^3 + xy^2 + k = c \]
leading to the same solution
\[ x^3 + xy^2 = c^* \]

Step 5. Check solution

\[ x^3 + xy^2 - c^* = 0 \]
\[ \frac{d}{dx} (x^3 + xy^2 - c^*) = 3x^2 + (y^2 + 2xy \frac{dy}{dx}) - 0 = 0 \]
\[ 3x^2 + y^2 \frac{dy}{dx} = 0 \]
\[ \frac{dy}{dx} = - \frac{(3x^2 + y^2)}{2xy}, = \text{ODE, checks} \]

**Application: time of travel** (related to particle tracking and residence time)

SOV and exact ODEs are two methods used to solve for the time of travel of a fluid parcel in hydrologic systems. For example, the parcel could represent a tracer particle or contaminant. The concept is applied in the atmosphere, oceans, surface water, and groundwater.

The basic idea is to identify a “flow line” along which a fluid parcel moves. Formally, this line is called a “path line”. If the flow is steady, that is, doesn’t change with time, the flow line has another name, “streamline”, which is a curve everywhere to which the instantaneous velocity is
tangent. During steady flow path lines and streamlines are the same. In the current situation let’s assume that while the strength (speed) of the flow field may change, and thus the flow is unsteady, the pattern of streamlines stays the same. This is a special case and is, of course, what always happens in one-dimension. An example of this in two dimensions is radial flow to or from a production/injection well (see plan view sketch). The flow pattern is always radial, but as the pumping/injection rate changes the flow changes speed and can reverse direction.

In this case of radial flow the speed changes with radius \( r \), due to geometry, and time \( t \), due to changing pumping/injection rate.

Let’s return to the general problem, where \( x \) is our coordinate system along the path line. What is the travel time of a fluid parcel, along a path line, from one point to another, say points \( A \) and \( B \)? This takes us back to kinematics and freshman physics. The relationship between location, \( x \) (or \( r \) in the radial flow example, above), and velocity, \( v \), is given by

\[
\frac{dx}{dt} = v(x,t) \quad \text{with IC} \quad x=x_0 \text{ at } t=t_0
\]

where \( x \) is the distance along the pathline, \( x_0 \) is the location of point \( A \), and we solve this IVP (initial value problem) to find the travel time to location \( B \).

What method do we use to solve this IVP?

Step 1. Is the ODE separable? That is, can we use SOV on \( v(x,t) \). The most trivial case, and one that is commonly assumed, is that \( v = \text{constant} \).

Step 1a. Is \( v \) constant? If so, then

\[
dx = v \, dt \quad \Rightarrow \quad \int dx = \int v \, dt = v \int dt \quad \Rightarrow \quad x = vt + c \quad \Rightarrow \quad x_0 = vt_0 + c \quad \Rightarrow \quad x - x_0 = v(t - t_0)
\]

The location of point \( B \) is then given by \( x_B = x_A + vt \), where I’ve assumed that the parcel passes point \( A \) at time \( t_0 = 0 \).

Step 2b. Is \( v(x,t) \) non-constant but separable? If so, then write it as a product, \( v = f(x) \, g(t) \)

\[
f(x)dx = g(t)dt \quad \Rightarrow \quad \int_{x_0}^{x} f \, dx = \int_{t_0}^{t} g \, dt
\]

There is another application. Suppose you are asked “how far can a fluid parcel travel in a prescribed time?” we can solve the same ODE for this, too.
To proceed further, we need the actual functions \( f \) and \( g \), and thus need to proceed with an example. Let’s take the radial flow problem from above. In that problem the velocity field is described by

\[
v = \frac{Q(t)}{2\pi rBn}
\]

where \( r = \) radius, \( n = \) porosity, \( B = \) aquifer thickness, \( Q(t) \) is time varying pumping/injection (\( Q \) is negative for pumping), and \( v(r,t) \) is the radial and time dependent “seepage velocity”. On the RHS, the numerator is the volumetric rate of water withdrawal or addition (\( \text{m}^3\text{s}^{-1} \)), while the denominator is the area through which fluid flux occurs, \( 2\pi rB \), at a radius \( r \), adjusted by effective porosity, \( n \).

We need to rewrite the time-of-travel model to accommodate the radial flow, or

\[
\frac{dr}{dt} = v(r,t) \quad \text{with IC} \quad r=r_0 \text{ at } t=t_0
\]

\[
v = f(r) \cdot g(t)
\]

\[
\frac{1}{f(r)} dr = g(t) \, dt \quad \Rightarrow \quad \int_0^r f^{-1} \, dr = \int_0^t g \, dt
\]

Let’s define \( f(r) \) and \( g(t) \) as

\[
f(r) = \frac{1}{r} \quad ; \quad g(t) = \frac{Q(t)}{2\pi Bn}
\]

To proceed further we need a function describing the pumping and injection, say,

\[
Q(t) = \bar{Q} + Q' \sin \omega t
\]

where \( \bar{Q} \) is the mean rate, \( Q' \) is the amplitude of pumping fluctuation, and \( \omega \) is the frequency (=2\( \pi/T \), where \( T \) is the period). If pumping matches injection, then \( \bar{Q} = Q' \), while if injection exceeds pumping (due to a desired for long-term storage in an ASR - aquifer storage and recovery - project), \( \bar{Q} > Q' \). In any event, the solution is

\[
\frac{r^2 - r_0^2}{2} = \frac{1}{2\pi Bn} \left[ \bar{Q}(t-t_0) + \frac{Q'}{\omega} (\cos \omega t_0 - \cos \omega t) \right]
\]

or, travel time \( t \), from radius \( r_0 \) to radius \( r \), is written implicitly as (for one cycle or less)

\[
r^2(t) = r_0^2 + \frac{1}{\pi Bn} \left[ \bar{Q}(t-t_0) + \frac{Q'}{\omega} (\cos \omega t_0 - \cos \omega t) \right]
\]

Step 3. If \( v(x,t) \) is not separable, then as if the ODE is an exact ODE, or if an integrating factor can be found to make it exact (not shown). If not, we go on to other methods …
Linear 1st Order ODEs (see text, §1.5, for proofs and details)

\[(1) \quad y' + p(x)y = r(x). \]

\[(2) \quad y' + p(x)y = 0 \quad \text{is called homogeneous.} \]

The general solution of the homogeneous ODE (2),

\[(3) \quad y(x) = ce^{-\int p(x) \, dx} \quad (c = \pm e^{\int p(x) \, dx} \quad \text{when} \quad y \geq 0); \]

here we may also choose \( c = 0 \) and obtain the trivial solution \( y(x) = 0 \) for all \( x \) in that interval.

Solution of nonhomogeneous linear ODE (1)

\[(4) \quad y(x) = e^{-h} \left( \int e^{h} r \, dx + c \right), \quad h = \int p(x) \, dx. \]

The structure of (4) is interesting. The only quantity depending on a given initial condition is \( c \). Accordingly, writing (4) as a sum of two terms,

\[(4*) \quad y(x) = e^{-h} \left( \int e^{h} r \, dx + ce^{-h} \right), \]

we see the following:

\[(5) \quad \text{Total Output = Response to the Input } \int r \, dx + \text{Response to the Initial Data.} \]

**First-Order ODE, Initial Value Problem**

Solve the initial value problem

\[y' + y \tan x = \sin 2x, \quad y(0) = 1.\]

**Solution.** Here \( p = \tan x, \ r = \sin 2x = 2 \sin x \cos x, \) and

\[\int p \, dx = \int \tan x \, dx = \ln |\sec x|.\]

From this we see that in (4),

\[e^h = \sec x, \quad e^{-h} = \cos x, \quad e^h r = (\sec x)(2 \sin x \cos x) = 2 \sin x, \]

and the general solution of our equation is

\[y(x) = \cos x \left( 2 \int \sin x \, dx + c \right) = c \cos x - 2 \cos^2 x. \]

From this and the initial condition, \( 1 = c \cdot 1 - 2 \cdot 1^2 \), thus \( c = 3 \) and the solution of our initial value problem is \( y = 3 \cos x - 2 \cos^2 x \). Here \( 3 \cos x \) is the response to the initial data, and \(-2 \cos^2 x \) is the response to the input \( \sin 2x \).
Hormone Level

Assume that the level of a certain hormone in the blood of a patient varies with time. Suppose that the time rate of change is the difference between a sinusoidal input of a 24-hour period from the thyroid gland and a continuous removal rate proportional to the level present. Set up a model for the hormone level in the blood and find its general solution. Find the particular solution satisfying a suitable initial condition.

Solution. Step 1. Setting up a model. Let \( y(t) \) be the hormone level at time \( t \). Then the removal rate is \( Ky(t) \). The input rate is \( A + B \cos \left(\frac{\pi t}{12}\right) \), where \( A \) is the average input rate, and \( A \geq B \) to make the input nonnegative. (The constants \( A, B, \) and \( K \) can be determined by measurements.) Hence the model is

\[
y''(t) = \text{In} - \text{Out} = A + B \cos \left(\frac{\pi t}{12}\right) - Ky(t) \quad \text{or} \quad y'' + Ky = A + B \cos \left(\frac{\pi t}{12}\right).
\]

The initial condition for a particular solution \( y_{\text{part}} \) is \( y_{\text{part}}(0) = y_0 \) with \( t = 0 \) suitably chosen, e.g., 6:00 A.M.

Step 2. General solution. In (4) we have \( p = K = \text{const}, h = Kt, \) and \( r = A + B \cos \left(\frac{\pi t}{12}\right) \). Hence (4) gives the general solution

\[
y(t) = e^{-Kt} \int e^{Kt} \left(A + B \cos \frac{\pi t}{12}\right) \, dt + ce^{-Kt}
\]

\[
= e^{-Kt} \frac{A}{K} + \frac{B}{144K^2 + \pi^2} \left(144K \cos \frac{\pi t}{12} + 12\pi \sin \frac{\pi t}{12}\right) + ce^{-Kt}.
\]

The last term decreases to 0 as \( t \) increases, practically after a short time and regardless of \( c \) (that is, of the initial condition). The other part of \( y(t) \) is called the steady-state solution because it consists of constant and periodic terms. The entire solution is called the transient-state solution because it models the transition from rest to the steady state. These terms are used quite generally for physical and other systems whose behavior depends on time.

Step 3. Particular solution. Setting \( t = 0 \) in (6) and choosing \( y_0 = 0 \), we have

\[
y(0) = \frac{A}{K} + \frac{B}{144K^2 + \pi^2} \cdot 144K + c = 0, \quad \text{thus} \quad c = -\frac{A}{K} - \frac{B}{144K^2 + \pi^2} \cdot 144K.
\]

Inserting this result into (6), we obtain the particular solution

\[
y_{\text{part}}(t) = \frac{A}{K} + \frac{B}{144K^2 + \pi^2} \left(144K \cos \frac{\pi t}{12} + 12\pi \sin \frac{\pi t}{12}\right) - \left(\frac{A}{K} + \frac{144KB}{144K^2 + \pi^2}\right) e^{-Kt}
\]

with the steady-state part as before. To plot \( y_{\text{part}} \), we must specify values for the constants, say, \( A = B = 1 \) and \( K = 0.05 \). Figure 17 shows this solution. Notice that the transition period is relatively short (although \( K \) is small), and the curve soon looks sinusoidal; this is the response to the input \( A + B \cos \left(\frac{\pi t}{12}\right) = 1 + \cos \left(\frac{\pi t}{12}\right) \).
A linear reservoir, lumped parameter model of an aquifer (JLW)

This is an exercise on ODE’s, flux, storage, and time constants. Refer to Example 3(Hormone Level) on p. 29 of the text, and on the previous page of these notes, for a homology.

The mathematics of this problem involves solutions to simple non-homogeneous, first-order, linear ODE’s with constant coefficients. While this model refers to an aquifer and a volumetric water balance, similar lumped-parameter models are ubiquitous and are used, e.g., at somewhat similar spatial scales (km’s) for the thermal energy (heat) balance in lakes and surface reservoirs like cooling ponds (intensive state = temperature), and at much smaller scales (mm) to represent the chemical mass balance in a feldspar grain due to intragranular diffusion and adsorption of the chemical (intensive state = concentration). This is also an introduction to concepts that apply in transient distributed parameter models.

Consider the following first order, linear ODE lumped parameter model of a ground-water aquifer water balance (there are similar lumped-parameter, volumetric water-balance models for lakes and surface reservoirs), [Gelhar, L.W. and J.L. Wilson, Ground-water quality modeling, Ground Water, 12, 399-408, 1974].

\[
S \frac{dh}{dt} = N - \alpha (h - h_o)
\]

where:

- \( h \) = mean water level in a phreatic aquifer, the dependent variable
- \( N \) = recharge rate [L/T],
- \( \alpha \) = outflow coefficient [1/T],
- \( S \) = storage coefficient [-],
- \( h_o \) = reference head [L],
- \( t \) = time [T], the independent variable.

Assume that parameter values are \( S = 0.1, N = 0.04 \text{ m/yr.}, \alpha = 0.004 \text{ /yr.}, h_o = 0 \text{ m}. \) The initial head is \( h = h_i = 1 \text{ m}. \) The datum is at the bottom of the aquifer, which is horizontal. The storage per unit area is (h-bottom) = h.

a. Solve this initial value problem (symbolically!) for and then plot \( h(t) \), for \( 0 \leq t \leq 150 \text{ years} \). What is the functional form of this solution? Use Matlab to plot the solution.

Solve (symbolically) for the flux in (\( = N \)) and flux out (\( = \alpha [h-h_o] \)), as a function of \( t \) and compare to storage per unit area (\( = h \)). Use Matlab to plot the three variables as a function of time.

The loose definition of a "time constant" \( t_h \) for an exponentially responding system (like a groundwater aquifer), is the time for the system state to make a substantial change. For changes between two (equilibrium) steady states the time constant is usually defined as the time it takes for the system state (\( h \)) to complete 63% of the change (leaving a \("1/e fold")))

80
change” left to the final steady state). Its related to another time constant definition, the so-called half life of the system \( H = t_1/2 \) (recall radioactive decay which is another exponentially responding system): \( t_1/2 = 0.693 \, t_h \). What is the time constant, \( t_h \), for this problem, both symbolically and in value (years)? What is the half life?

\( b. \) The initial condition in part \( a \) is "out of balance" with the recharge rate and the outflow, and thus we get a transient response, \( h(t) \). Suppose instead that the recharge rate is changing slowly with time, according to the model:

\[
N = \begin{cases} 
\bar{N}, & t \leq 0 \\
\bar{N} + N' \sin(\omega t), & t > 0 
\end{cases}
\]

where:

- \( \bar{N} \) = mean recharge rate (0.04 m/yr.),
- \( N' \) = recharge rate amplitude (0.03 m/yr.),
- \( \omega \) = frequency of recharge fluctuation,
  \[ \omega = 2\pi / T, \]
- \( T \) = recharge fluctuation period (yrs).

Let \( T = 1 \) year, reflecting seasonal variability in recharge. Solve, plot (fluxes and storage), and study this problem for \( 0 \leq t \leq 150 \) years. Repeat with \( T = 25 \) years, reflecting decadal periods of drought and 'flood'. Finally, repeat with \( T = 500 \) years reflecting even longer term climate changes. Be prepared to discuss. It is useful to consider the ratio of period to aquifer time constant \( T/t_h \) in your study and discussion. Why is this ratio useful when characterizing behavior?

Note: in the limit you’ll find that the solution becomes quasi-steady, when the initial condition is no longer important \( (t >> t_h) \). In this context (as \( t \to \infty \)) a plot of dimensionless input (and response) frequency versus dimensionless response amplitude is very useful. Attempt to create this plot using Matlab. You may want to consider if you should use log of dimensionless time in the plot. Using this plot explain how an aquifer acts like a “low-pass filter.” In particular, explain why system response becomes simpler to model and understand if the period of forcing is much smaller or much longer than the aquifer time constant, and how and why the simplification for these two special conditions is different.

\( c. \) Add Pumping: Suppose you were to examine a model and parameters similar to part \( a \) of this exercise, but with an additional flux (a sink) due to pumping well extraction, and a different IC. The new model is

\[
S \frac{dh}{dt} = N - W - \alpha (h - h_o)
\]

where \( W \) [L/T] is the (per unit area) pumping rate, \( N \) is constant natural recharge, and all other terms are as before. The parameter values are \( S = 0.1, \, N = 0.04 \) m/yr., \( \alpha = 0.004/\)yr., \( h_o = 0 \) m. Assume that the initial head, \( h(t=0) = h_i \) and the discharge, \( \alpha (h-h_o) \) are in (steady state)

\[
81
\]
equilibrium with the constant natural recharge, \( N \), and there is no pumping for \( t<0 \). Along comes a real estate development pumping at rate \( W \), but because of population growth, \( W(t) \) grows exponentially, starting out at some initial rate \( W(t=0)=W_i \).

i. What is the initial steady state solution for \( h(t\leq 0) \) (symbolically)?

ii. What is the transient solution to (4) if \( W \) is initially \( (t=0) \) given by \( W_i = 0.01 \text{ m/yr} \) and thereafter grows exponentially with a doubling time of 10 years. You'll have to convert doubling time to the value of the parameter \( t_w \) in the function \( W=W_i \exp(t/t_w) \). Solve symbolically and plot the result using Matlab. NOTE: this is a new load and requires a new solution; you can’t use (1) or the solution from Part b of the exercise.

iii. Determine symbolically how long this pumping scheme will last before \( h \) becomes smaller than \( h_o \), and the discharge from the aquifer becomes recharge back to the aquifer from the adjacent source, \( h_o \). That is, at what time does \( h(t)=h_o \)?

iv. Discuss your answers to a, b, c in terms of fundamental behavior and properties, such as the aquifer and pumping growth time constants.

NOTE: Use error-checking guidance to ensure that you have correct solutions. For example, suppose you did this problem (a, b or c) the wrong way, how would you realize your error?

Three common error checks (but by no means all of the possible error checks), in the order you would normally apply them are: first, a dimensional check; second a check would see if the solution preserves the initial condition; and third, you would substitute your solution for \( h(t) \) into the original model, for part a this is

\[
S \frac{dh}{dt} = N - \alpha(h - h_o),
\]

(3)

to see if it satisfies the ODE. A fourth check is to query your answer to see if it makes sense.
Example application of a linear reservoir groundwater model in a modern commercial code used for watershed studies

“In MIKE BASIN, a groundwater storage (aquifer) is conceptualized as a linear reservoir with one (default) or two (optional) layers:

\[
\frac{\partial h_1}{\partial t} = (-k_1 - k_2)(h_1 - L_1) + q_{\text{recharge}} + q_{\text{stream seepage}} \\
\frac{\partial h_2}{\partial t} = k_2(h_1 - L_1) - k_3(h_2 - L_2) - q_{\text{pumping}}
\]

where the variables are defined in the figure. The dimensions of \( L \) and \( h \) are \([\text{L}]\). Note that an (outflow) rate constant \( k \) \([\text{1/T}]\) is the inverse of the outflow time constant that is specified in the catchment node dialog. The fluxes \( q \) are area-specific \([\text{L/T}]\).

The mathematical solution for the simplest linear reservoir model, one without inflows, is an exponential decay in storage with time. A time constant \( t \) determines the speed of the exponential decay. For the simplest linear reservoir model, after time \( t \), 36.8% \((1/e)\) of the original storage remain. With simultaneous inflow into the storage, drainage can still be computed analytically, but the time constant \( t \) will not be so directly related to a particular percentage of remaining storage.

It is possible for a groundwater storage to be emptied (when outflows permanently exceed inflows). Also overflow is possible (when inflows permanently exceed outflows). Finally, another special case is when the lower groundwater level reaches the shallow outlet, causing flow back into the shallow reservoir. The mathematical solution of the linear reservoir equations in MIKE BASIN is valid also for all those situations. Water quality in groundwater is explained separately, but the explanation also refers to the above figure.”

http://www.dhisoftware.com/mikebasin/Description/
http://www.dhisoftware.com/mikebasin/helpsys/mbasinLinear_Reservoir.htm
Reduction of a Non-Linear Model to Linear Form

Many ODEs can be transformed (exactly) to linear form through simple transformation relationships. In your text reference [A11] lists many common examples of this. In hydrology these transformations are most common in vadose zone hydrology.

Note: Typically the non-linearity enters via the dependence of properties (like unsaturated permeability) on state (the dependent variable, like pressure or moisture content). By selecting an appropriate, but simple functional model of that relationship, the model can be exactly linearized. The approximation comes from fitting a convenient but possibly poor model to the property curves.

Numerous applications can be modeled by ODEs that are nonlinear but can be transformed to linear ODEs. One of the most useful ones of these is the Bernoulli equation

\[ y' + p(x)y = g(x)y^a \quad (a \text{ any real number}). \]

If \( a = 0 \) or \( a = 1 \), Equation (6) is linear. Otherwise it is nonlinear. Then we set

\[ u(x) = [y(x)]^{1-a}. \]

We differentiate this and substitute \( y' \) from (6), obtaining

\[ u' = (1 - a)y^{-a}y' = (1 - a)y^{-a}(gy^a - py). \]

Simplification gives

\[ u' = (1 - a)(g - py^{1-a}), \]

where \( y^{1-a} = u \) on the right, so that we get the linear ODE

\[ u' + (1 - a)pu = (1 - a)g. \]

Logistic Equation

Solve the following Bernoulli equation, known as the logistic equation (or Verhulst equation):

\[ y' = Ay - By^2 \]

Where \( A \) and \( B \) are constants. Then applying these methods … (see text)

\[ y - \frac{1}{u} - \frac{1}{ce^{-At} + B/A} \quad \text{ (Eq. 18)}. \]

Directly from (8) we see that \( y = 0 \) (\( y(t) = 0 \) for all \( t \)) is also a solution.

Fig. 18. Logistic population model. Curves (9) in Example 4 with \( A/B = 4 \).
Orthogonal Trajectories

What are orthogonal trajectories?
A family of curves that intersect at right angles. In many cases they can be found by solving odes, as shown in the text.

Why do we care?
There are many applications of this concept in hydrology: contour lines of equal elevation on a topographic map and directions of steepest descents, streamlines and equipotentials in a potential flow field, flow lines and lines of equal head in an aquifer, curves of heat flow and lines of equal temperature, etc … The physics of the problem has to have certain characteristics for this to be true, but as you can see from all these examples it is not uncommon. For fluid flow a measure of the appropriateness of this concept are the Cauchy Riemann conditions (see problem 15 on page 37). Flow fields that meet this condition are potential flow fields and have orthogonal streamlines and equipotentials.

Examples

Fig. 24. Flow in a channel in Problem 16
This corner flow field illustrates orthogonal streamlines and equipotentials. The corner itself represents a singularity in which the speed goes to zero. This type of singularity is called a stagnation point.

Stagnation points are interesting for a variety of reasons. For example, from Bernoulli’s equation we know that they have zero kinetic energy. Instruments are designed around this principle. Also, if a solute or colloid approaches a stagnation point it slows down, and eventually (in the limit) can stop. Thus stagnation zones are zones where chemicals and colloids can accumulate. On a large, hydrogeologic scale it is believed that some oil reservoirs in southeastern New Mexico may be due to hydrodynamic traps created by

Fig. 25. Electric field in Problem 18

Fig. 25 is also the solution for streamlines and equipotentials near a point source/sink pair in a 2D potential flow field.

Example application is the 2D flow field between an aquifer remediation injection-production well pair.

Another application is the use of superposition to represent a pumping well near a stream (where the head is assumed constant).

Note that the point sink in Fig. 22, and the point sources/sinks in this figure, are singularities. In this case multiple streamlines meet at the sources/sinks, and speeds go to infinity.

Aside: Singularities give numerical methods fits. Mechanical and civil engineering numerical codes, which are usually based on the finite element approach, use special elements. These elements invoke analytical solutions for conditions near singularities. Typically these include the conditions in Figs. 22 and 24. Some groundwater models do the same for conditions near wells. Mostly, however, one simply increases the numerical grid density near singularities.
Existence and Uniqueness of Solutions

Motivation:
Initial value (IV) problems, consisting of an ODE and IC,

\[ y' = f(x, y), \quad y(x_0) = y_0 \]  \hspace{1cm} (1)

can result in
- no solution (or only a trivial solution, \( y(x) = 0 \)),
- one solution, or
- many or infinitely many solutions.

**Existence:** Under what conditions does an IV problem of the form (1) have at least one solution (and hence one or several solutions)?

Conditions for existence:

**Existence Theorem**

Let the right side \( f(x, y) \) of the ODE in the initial value problem

\[ y' = f(x, y), \quad y(x_0) = y_0 \]  \hspace{1cm} (1)

be continuous at all points \( (x, y) \) in some rectangle

\[ R: |x - x_0| < a, \quad |y - y_0| < b \]  \hspace{1cm} (Fig. 26)

and bounded in \( R \); that is, there is a number \( K \) such that

\[ |f(x, y)| \leq K \quad \text{for all} \; (x, y) \in R. \]  \hspace{1cm} (2)

Then the initial value problem (1) has at least one solution \( y(x) \). This solution exists at least for all \( x \) in the subinterval \( |x - x_0| < \alpha \) of the interval \( |x - x_0| < a \); here, \( \alpha \) is the smaller of the two numbers \( a \) and \( b/K \).

That is, if \( f(x, y) \) is continuous in some region of the \( x, y \)–plane containing the point \( x_0, y_0 \), then the IV problem (1) has at least one solution.

Existence is the focus of many mathematical studies. We need to be aware of this issue, particularly that \( f(x, y) \) need be continuous, but don’t need to focus on proofs. It is unlikely that you’ll encounter many problems for which an existence proof does not already exist. However, if you are posing a new problem, it is something you’ll have to worry about. Consult a mathematician if you can’t figure it out yourself.

However, the uniqueness issue is of particular importance to us, as we have many forward problems with unique solutions, but few inverse problems that are unique.
**Uniqueness**: Under what conditions does an IV problem of the form (1) have at most one solution (hence excluding the case that it has more than one solution).

**Conditions for uniqueness:**

Uniqueness Theorem

Let \( f \) and its partial derivative \( f_y = \frac{\partial f}{\partial y} \) be continuous for all \((x, y)\) in the rectangle \( R \) (Fig. 26) and bounded, say,

\[
(a) \quad |f(x, y)| \leq K, \quad (b) \quad |f_y(x, y)| \leq M \quad \text{for all } (x, y) \text{ in } R.
\]

Then the initial value problem (1) has at most one solution \( y(x) \). Thus, by Theorem 1, the problem has precisely one solution. This solution exists at least for all \( x \) in that subinterval \( |x - x_0| < \alpha \).

That is, if the partial derivative, \( \partial f / \partial y \), of \( f(x,y) \) with respect to \( y \) exists and is continuous in some region of the \( x,y \)–plane containing the point \( x_0,y_0 \), then the IV problem (1) has at most one solution; hence by the existence theorem it has precisely one solution.