GEOP 505/MATH 587 – Homework 6

The following homework is due on Monday, November 28, by 5pm.

1. Consider the Kalman filtering problem for a system with extremely simple dynamics. The state vector is of size 1 by 1. The state, $x_k$, is constant, and there is no random perturbation of the state. The measurements $z_k$ are direct measurements of the state, with errors that are normally distributed with mean 0 and standard deviation $\sigma$. We begin by using the first measurement, $\hat{x}_0 = z_0$.

(a) Formulate this as a Kalman filtering problem. What are $A$, $B$, $H$, $Q$, and $R$?

(b) What are $\hat{x}_k^-$ and $\hat{P}_k^-$?

(c) What is the optimal Kalman gain for this problem?

(d) Find a simple formula $\hat{P}_k$ in terms of $\sigma$.

(e) Find a simple formula for $\hat{x}_k$ in terms of the observations $z_0, z_1, \ldots$.

(f) How do your results in parts (d) and (e) compare with the common approach to estimating $x$ and obtaining a “standard error of the mean” from $n$ independent and normally distributed measurements?

2. Consider a spring–mass system governed by the second order differential equation

$$my''(t) + cy'(t) + ky(t) = F(t)$$

where $m = 1$ Kg, $c = 2$ N s/m, $k = 2$ N/m, and $F(t) = 2\sin(5t)$ N. Our best guess of the initial state of the system is that $y(0) = 0.1$ m, with a one-\(\sigma\) uncertainty of 0.05 m, and $y'(0) = 1$ m/s, with a one-\(\sigma\) uncertainty of 0.5 m/s. At times $t = 0.5, 1.0, 1.5$, we observe that $y(0.5) = 0.58$ m, $y(1.0) = 0.63$ m, and $y(3) = 0.31$ m. All of these measurements have one-\(\sigma\) uncertainties of 0.05 m.

(a) Convert this second order ODE into a system of 2 first order linear ordinary differential equations.

(b) Use Euler’s method

$$x(t + \Delta t) \approx x(t) + \Delta t x'(t)$$

to discretize the system of differential equations using time steps of $\Delta t = 0.001$ seconds. Write your discrete time dynamical system in the form

$$x_k = Ax_{k-1} + Bu_{k-1}.$$ 

Also determine the observation equation. What are $H$, $Q$, and $R$?
(c) Ignoring the observations, and using only the prediction step of the Kalman filter, predict the position of the mass at times $k = 0, 1, 2, \ldots, 2001$ (that is, for the first two seconds.) Plot your prediction of the position and velocity of the mass with associated one-$\sigma$ error bars. What is your predicted position at time $t = 2$ seconds?

(d) Now, repeat the prediction using the three observations. Plot your prediction with with associated error bars. What is your predicted position at time $t = 2$ seconds? How much do these observations improve the prediction?