FEM Method

We will explore:

1-D linear & higher order elements
2-D triangular & rectangular elements

Powerful method developed originally to solve structural mechanics problems (e.g. bridges, buildings, etc..)

Can be applied to flow, solute, and heat transport problems

FEM Method

• Has underlying theoretical basis in branch of mathematics called calculus of variation
• Within each element, the unknown variable is approximated using polynomials that depend only on the spatial coordinates (i.e. \( h = a + bx + cy + \ldots \)). The vertices of the elements are referred to as nodes.
• Originally developed for solid mechanics problems

FEM Mesh of Baseball and Bat

Fig. 9 Sequence of ball deformation during contact with bat/rotor

Triangular Element

Fig. 10 Profile of the aluminum bat model

FEM Mesh of Dam
Finite Element Method & Calculus of Variations

- Involves minimizing residual errors using integration
- Conceptually similar to the cannon ball trajectory problem solved using the calculus of variations method (see Feynman's lecture notes)

\[
Action = S = \int_0^t [KE - PE]dt = \int_0^t \left[ \frac{1}{2} m \left( \frac{dx}{dt} \right)^2 - mgx \right] dt = \int_0^t [F(x,t)]dt
\]

\[
x(t) = \bar{x}(t) + \eta(t)
\]

\[
\int_0^t \left[ \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + mg \frac{dx}{dt} \right] \eta(t) dt = \int_0^t [F(x,t)]\eta(t) dt = 0
\]

- FD method using Taylor's Series Expansion
- FE method uses Integration by Parts

\[
\int u dv = uv - \int v du
\]

Example: \[
\int xe^x dx
\]

Let \( e^x dx = dv \) \quad \therefore \quad v = e^x

Let \( u = x \) \quad \therefore \quad du = dx

\[
\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x
\]
Steps in FEM Method, 1D Example

• Formulate Diff. Eq.\[
\frac{\partial}{\partial x} \left( T \frac{\partial h}{\partial x} \right) = Q
\]

\( T \) – transmissivity (m²/day)  
\( H \) – head (m)  
\( Q \) – recharge (m/day)

\( h(0,t) \) = specified head  
\( q(0,t) \) = specified flux

• Discretize Solution Domain

• Define Local Coordinate System

\[ 0 \leq \bar{x} \leq \Delta x \]  
\[ \Delta x = x_2 - x_1 \]

Steps in FEM Method, 1D Example

• Construct (linear) Trial Solution for each element

\[ \hat{h} = a + b\bar{x} \]  
\[ h_1 = a \]  
\[ h_2 = a + b\Delta x \]

\[ b = \frac{h_2 - h_1}{\Delta x} \]  
\[ \hat{h} = [h_1 + h_2 - h_1 \bar{x}]/\Delta x \]

\[ \hat{h} = \left[ h_1 \right] - \left[ \frac{\bar{x}}{\Delta x} \right] + \left[ \frac{\bar{x}}{\Delta x} \right] = \sum_{n=1}^{\triangle} \phi_n h_n \]

\[ \phi_1 = \left[ 1 - \frac{\bar{x}}{\Delta x} \right] \]  
\[ \phi_2 = \left[ \frac{\bar{x}}{\Delta x} \right] \]  
\( \phi \) - Shape functions
Steps in FEM Method, 1D Example

• Construct (linear) Trial Solution for each element

\[ \hat{h} = a + b\bar{x} \quad h_1 = a \quad h_2 = a + b\Delta x \]

\[ b = \frac{h_2 - h_1}{\Delta x} \quad \hat{h} = h_1 + \frac{h_2 - h_1}{\Delta x} \bar{x} \]

\[ \hat{h} = h_1 \left[ 1 - \frac{\bar{x}}{\Delta x} \right] + h_2 \left[ \frac{\bar{x}}{\Delta x} \right] = \sum_{n=1}^{2} \phi_n h_n \]

\[ \phi_1 \left[ 1 - \frac{\bar{x}}{\Delta x} \right] \quad \phi_2 = \frac{\bar{x}}{\Delta x} \quad \phi \cdot \text{Shape functions} \]

Shape Function Properties

• Linear variation across element from 0 to 1

• Derivatives of linear shape functions are constant

\[ \frac{\hat{h}}{\Delta x} = \sum_{n=1}^{2} \frac{\partial \phi_n}{\partial \bar{x}} h_n \]

\[ \frac{\partial \phi_1}{\partial \bar{x}} = \frac{1}{\Delta x} \quad \frac{\partial \phi_2}{\partial \bar{x}} = \frac{1}{\Delta x} \]
1D-FEM

- Substitute Trial Solution into Diff. Eq.
  \[ \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) - Q \neq 0 \]

- Multiply trial solution by a weighting function and require the weighted residual errors integrate to zero across the element
  \[ \int_{0}^{L} \left[ \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) - Q \right] v dx = 0 \]

\[ v = \sum_{m=1}^{N} \phi_m \]
\[ \sum_{m=1}^{N} \phi_m(x) = 1 \]

1D-FEM Steps

- Use Integration by parts to “lower the order of the PDE”
  \[ \int_{0}^{L} \left[ \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) - Q \right] v dx = - \int_{0}^{L} v \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) dx - \int_{0}^{L} Q v dx + \int_{0}^{L} v \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) dx = 0 \]

- Substitute Trial Solution and weighting function into “weak form of PDE”
  \[ v = \text{weighting Function} \]
  \[ h = \sum_{N=1}^{2} \Phi_N(\bar{x}) h_N \]
  \[ v = \sum_{M=1}^{2} \Phi_M(\bar{x}) \]

\[ \bar{x} \]
1D-FEM Steps

- This results in a system of algebraic Equations

\[
\sum_{m=1}^{2} \sum_{n=1}^{2} \int_0^{\Delta x} T \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_n}{\partial x} dx - \sum_{m=1}^{2} \int_0^{\Delta x} Q \phi_m dx = 0
\]

\[A_{mn} h_n = B_m\]

\[
A_{mn} = \sum_{m=1}^{2} \sum_{n=1}^{2} \int_0^{\Delta x} T \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_n}{\partial x} dx
\]

\[B_n = \sum_{m=1}^{2} \int_0^{\Delta x} Q \phi_m dx = 0\]

1D-FEM Steps

- Evaluate Integrals: \(m=1, n=1\)

\[
A_{11} = A_{22} = \int_0^{\Delta x} T \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_1}{\partial x} dx = \int_0^{\Delta x} T \frac{\partial \phi}{\partial x} dx = \frac{T}{\Delta x}
\]

\[
B_1 = \int_0^{\Delta x} Q \phi dx = \int_0^{\Delta x} Q \left[ 1 - \frac{x}{\Delta x} \right] dx = \int_0^{\Delta x} Q \left[ 1 - \frac{x^2}{2\Delta x^2} \right] = Q \frac{\Delta x}{2}
\]

- Evaluate Integrals: \(m=2, n=1\)

\[
A_{21} = A_{12} = \int_0^{\Delta x} T \frac{\partial \phi_2}{\partial x} \frac{\partial \phi_1}{\partial x} dx = \int_0^{\Delta x} T \frac{\partial \phi_2}{\partial x} dx = -\frac{T}{\Delta x}
\]

\[
B_2 = \int_0^{\Delta x} Q \phi_2 dx = \int_0^{\Delta x} Q \left[ \frac{x}{\Delta x} \right] dx = \int_0^{\Delta x} Q \left[ \frac{x^2}{2\Delta x^2} \right] = Q \frac{\Delta x}{2}
\]
Properties of Shape Functions:

\[ \sum_{n=1}^{2} \Phi_n(x) = \Phi_1 + \Phi_2 = 1 \]

\[ \Phi_1(x = x_2) = \Phi_2(x = x_1) = 0; \quad \text{or} \]

\[ \Phi_n(x) = \delta_n \]

This is true for any point along the element (i.e. \( 0 \leq x \leq \Delta x \)).

1D-FEM

Properties of Shape Functions:

\[ \frac{\hat{h}}{\Delta x} = \sum_{n=1}^{2} \frac{\partial \Phi_n}{\partial x} h_n = \frac{\partial \Phi_1}{\partial x} h_1 + \frac{\partial \Phi_2}{\partial x} h_2 \]

\[ \frac{\hat{h}}{\Delta x} = \frac{h_2}{\Delta x} + \frac{h_1}{\Delta x} \]
FEM-1D Steps

\[ A_{\text{element-1}} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \]
\[ B_{\text{element-2}} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} h_s \\ h_2 \\ \Delta x \end{bmatrix} = \frac{\Delta x}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]
\[ \begin{bmatrix} T \\ \Delta x \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} h_s \\ h_2 \\ \Delta x \end{bmatrix} = \frac{\Delta x}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

Assemble Global Matrix:

\[ \begin{bmatrix} T \\ \Delta x \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} h_s \\ h_2 \\ \Delta x \end{bmatrix} = \frac{\Delta x}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \]

Steps 1D FEM

- Impose specified Head Boundary Condition @ x=0

\[ h(x=0) = h_s \]

\[ \begin{bmatrix} T \\ \Delta x \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} h_s \\ h_2 \\ \Delta x \end{bmatrix} = \frac{\Delta x}{2} \begin{bmatrix} h_s \\ q + \frac{\Delta x}{2} \end{bmatrix} \]

- Impose specified Flux Boundary Condition @ x=L

\[ q(x=L) = q_s \]

\[ \begin{bmatrix} T \\ \Delta x \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} h_s \\ h_2 \\ \Delta x \end{bmatrix} = \frac{\Delta x}{2} \begin{bmatrix} h_s \\ q + \frac{\Delta x}{2} \end{bmatrix} \]

Generalized Tridiagonal Matrix…look familiar??
1D FEM Steps

• Let:
  \[ T^1 = T^2 = \frac{1\text{m}^2}{\text{day}} \quad \Delta x^1 = 1\text{m} \quad \Delta x^2 = 2\text{m} \]
  \[ Q^1 = Q^2 = \frac{6\text{m}}{\text{day}} \]
  \[ q_o = 1 \]
  \[ h_o = 0 \]

• Global Matrix Becomes:
  \[
  \begin{bmatrix}
  T & 0 & 0 \\
  0 & 1.5 & -0.5 \\
  0 & -0.5 & 0.5 \\
  \end{bmatrix}
  \begin{bmatrix}
  h_1 \\
  h_2 \\
  h_3 \\
  \end{bmatrix}
  =
  \begin{bmatrix}
  0 \\
  9 \\
  7 \\
  \end{bmatrix}
  \]

• Solution (Gaussian Elimination):
  \[
  \begin{bmatrix}
  h_1 \\
  h_2 \\
  h_3 \\
  \end{bmatrix}
  =
  \begin{bmatrix}
  0 \\
  16 \\
  30 \\
  \end{bmatrix}
  \]

Analytical Solution Comparison

\[ h = h_o - \frac{Q}{T} \left[ \frac{x^2}{2} - xL \right] + \frac{q_o x}{T} \]

Unlike the finite difference method, the finite element method provides us with an estimate of the hydraulic heads everywhere within the solution domain.
Determination of Elemental Fluxes

FEM: \[ q = T \frac{\partial \hat{h}}{\partial x} = \sum_{N=1}^{2} \frac{\partial \Phi_i}{\partial x} h_N = T \left[ \frac{\partial \Phi_1}{\partial x} h_1 + \frac{\partial \Phi_2}{\partial x} h_2 \right] \]

\[ q = -T \left[ \frac{h_1}{\Delta x^2} + \frac{h_2}{\Delta x^2} \right] \]

Analytical: \[ q = +Q[x - L] - q_0 \]

Higher Order Elements

\[ \hat{h} = a + b\bar{x} + c\bar{x}^2 \quad h_1 = a \quad h_2 = a + b \frac{\Delta x}{2} + c \left( \frac{\Delta x}{2} \right)^2 \quad h_3 = a + b\Delta x + c\Delta x^2 \]

\[ \hat{h} = \sum_{n=1}^{3} \Phi_n h_n \]

\[ \Phi_1 = 1 - \frac{3x}{\Delta x} + 2 \left( \frac{x}{\Delta x} \right)^2 \quad \Phi_2 = 4 \frac{x}{\Delta x} \left( 1 - \frac{x}{\Delta x} \right) \quad \Phi_3 = -\frac{x}{\Delta x} \left( 1 - 2 \frac{x}{\Delta x} \right) \]
Lagrange Family of Shape Functions

\[ \phi_i = \frac{\prod_{j=1}^{N} (x - x_j)}{\prod_{j=1}^{N} (x_i - x_j)} \quad i \neq j \]

Example 1: N=2 (linear element)

\[ \phi_1 = \frac{(x - x_2)}{(x_1 - x_2)} = \frac{(x - \Delta x)}{(0 - \Delta x)} = 1 - \frac{x}{\Delta x} \]

Example 2: N=3 (quadratic element)

\[ \phi_i = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x - \Delta x)(x - \Delta x)}{\Delta x^2} = x^2 - \frac{\Delta x^2}{2} - 3x \frac{\Delta x}{2} \]

\[ \phi_3 = 1 - 3x \frac{\Delta x}{2} \cdot \left( \frac{x}{\Delta x} \right)^2 \]

Properties of Lagrange Shape Functions:

\[ \sum_{i=1}^{N} \phi_i = 1 \]

\[ \phi_i(x_j) = 0 \quad i \neq j \]

\[ \phi_i(x_i) = 1 \]
Derivatives of Shape Functions

\[ \frac{\partial h}{\partial x} = \sum_{n=1}^{3} \frac{\partial \phi_n}{\partial x} h_n \]

\[ \Phi_1 = 1 - \frac{3x}{\Delta x} + 2\left(\frac{x}{\Delta x}\right)^2 \quad \Phi_2 = 4 \frac{x}{\Delta x} \left(1 - \frac{x}{\Delta x}\right) \quad \Phi_3 = -\frac{x}{\Delta x} \left(1 - 2 \frac{x}{\Delta x}\right) \]

\[ \frac{\partial \Phi_1}{\partial x} = -\frac{3}{\Delta x} + \frac{4x}{\Delta x^2} \quad \frac{\partial \Phi_2}{\partial x} = 4 \frac{1}{\Delta x} - \frac{8x}{\Delta x^2} \quad \frac{\partial \Phi_3}{\partial x} = -\frac{1}{\Delta x} + \frac{4x}{\Delta x^2} \]

1D-FEM-Quadratic

- Substitute Trial Solution into Diff. Eq.
  \[ \frac{\partial}{\partial x} \left[ \int \frac{\partial h}{\partial x} \right] - Q \neq 0 \]

- Multiply trial solution by a weighting function and require the weighted residual errors integrate to zero across the element
  \[ \int_0^L \left[ \frac{\partial}{\partial x} \left( \frac{\partial h}{\partial x} \right) - Q \right] v \, dx = 0 \]

  \( v \) – weighting Function

\[ v = \sum_{n=1}^{3} \phi_n \]

\[ \sum_{n=1}^{3} \Phi_n(x) = 1 \]
1D-FEM Quadratic Steps

- This results in a system of algebraic Equations

\[ \sum_{m=1}^{3} \sum_{n=1}^{3} \int_{0}^{\Delta x} T \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_n}{\partial x} dx - \sum_{m=1}^{3} \int_{0}^{\Delta x} Q \phi_m dx = 0 \]

\[ A_{mn} h_n = B_m \]

\[ A_{mn} \sum_{m=1}^{3} \sum_{n=1}^{3} \int_{0}^{\Delta x} T \frac{\partial \phi_m}{\partial x} \frac{\partial \phi_n}{\partial x} dx \]

\[ B_m = \sum_{m=1}^{3} \int_{0}^{\Delta x} Q \phi_m dx = 0 \]

Evaluate Quadratic \( A_{mn} \)

- Evaluate Integrals: m=1, n=1

\[ A_{11} = \int_{0}^{\Delta x} T \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_1}{\partial x} dx = \int_{0}^{\Delta x} \left[ \frac{3}{\Delta x} + 4x \right] dx = \int_{0}^{\Delta x} \left[ \frac{9}{\Delta x^2} + \frac{24x}{\Delta x} + \frac{16x^2}{\Delta x} \right] dx \]

\[ A_{11} = \frac{9\Delta x}{\Delta x^2} - \frac{12\Delta x^2}{\Delta x^3} + \frac{16\Delta x^3}{3\Delta x^4} \]

\[ A_{11} = \frac{7\Delta x}{3\Delta x} \]

- Evaluate Integrals: m=2, n=1

\[ A_{21} = A_{11} = \int_{0}^{\Delta x} T \frac{\partial \phi_1}{\partial x} \frac{\partial \phi_2}{\partial x} dx = \int_{0}^{\Delta x} \left[ \frac{3}{\Delta x} + 4x \right] dx \]

\[ A_{21} = \int_{0}^{\Delta x} \left[ \frac{12}{\Delta x^2} + \frac{16x}{\Delta x^2} + \frac{24x}{\Delta x} - \frac{32x^2}{\Delta x^2} \right] dx \]

\[ A_{21} = \int \left[ \frac{12\Delta x}{\Delta x^2} + \frac{8\Delta x^2}{\Delta x^3} + \frac{12\Delta x^2}{\Delta x^2} - \frac{32\Delta x^3}{3\Delta x^4} \right] dx = \frac{8\Delta x}{3\Delta x} \]

You can do the rest....
Load Vector

\[ B_1 = \int_{0}^{h} Q \Phi_1 dx = \int Q \left[ 1 - \frac{3x}{\Delta x} + 2 \left( \frac{x}{\Delta x} \right)^2 \right] dx = Q \left[ \Delta x - \frac{3\Delta x^2}{2\Delta x} + \frac{2\Delta x^3}{3\Delta x^2} \right] = \frac{Q\Delta x}{6} \]

\[ B_2 = \int_{0}^{h} Q \Phi_2 dx = \int Q \left[ 4 \frac{x}{\Delta x} \left( 1 - \frac{x}{\Delta x} \right) \right] dx = \int Q \left[ \frac{4x}{\Delta x} - \frac{4x^2}{\Delta x^2} \right] dx = Q \left[ \frac{2\Delta x^2}{\Delta x} - \frac{4\Delta x^3}{3\Delta x^2} \right] = \frac{2Q\Delta x}{3} \]

You can do \( B_3 \)

**Final \( A_{mn} \) Matrix**

**Quad. Element**

\[
A_{mn} = \frac{T}{3\Delta x} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}
\]

**Linear Element**

\[
\begin{bmatrix} T & -1 \\ \Delta x & -1 & 1 \end{bmatrix}
\]

**Final \( B_m \) Vector**

\[
B_m = \frac{Q\Delta x}{6} \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}
\]

\[
\begin{bmatrix} Q\Delta x \\ 2 \end{bmatrix}
\]
Assemble Global Matrix

Note $A_{mn}$ Matrix is no longer tridiagonal

Let:
$T$(element-1) = 3
$T$(element-2) = 3
$\Delta x = 1$
$q_o = 1$
$h_o = 0$
$Q$(element-1) = 6
$Q$(element-2) = 6
$h_1 = 0$
$h_2 = 1.91$
$h_3 = 3.333$
$h_4 = 4.25$
$h_5 = 4.667$
Solve System of Equations

\[ q^1 = -3 \left[ b_1 \frac{\Delta x}{\Delta x^2} + b_2 \frac{\Delta x}{\Delta x^2} \right] + b_3 \left[ \frac{\Delta x}{\Delta x^2} + \frac{4 \Delta x}{\Delta x^2} \right] \]

(Use local coordinate system
For elemental flux calc. (0<x<\Delta x)

\[ q^2 = -3 \left[ b_1 \frac{\Delta x}{\Delta x^2} + b_2 \frac{\Delta x}{\Delta x^2} \right] + b_3 \left[ \frac{\Delta x}{\Delta x^2} + \frac{4 \Delta x}{\Delta x^2} \right] \]

Transient FEM Problems

\[ \frac{\partial}{\partial x} \left[ \mathbf{T} \frac{\partial h}{\partial x} \right] = \mathbf{S} \frac{\partial h}{\partial t} \]

The finite element solution to this time dependent problem is very similar to the approach taken to solve Poisson’s Equation. We use the same trial solutions:

\[ \hat{h} = \sum_{n=1}^{2} \phi_h \cdot h_n \]

\[ \phi_1 = \left[ 1 - \frac{x}{\Delta x} \right] \quad \phi_2 = \frac{x}{\Delta x} \]
Transient FEM Problems

\[ \int_0^x vS \frac{\partial h}{\partial t} dx + \int_0^x T \frac{\partial}{\partial x} \frac{\partial h}{\partial x} dx = 0 \]

Since \( h_n \) is not a function of "t", we get the following integrand

For the time dependent derivative:

\[ \int_0^x vS \frac{\partial h}{\partial t} dx = S_j \frac{\partial h_j}{\partial t} \sum_{n=1}^{2} \sum_{m=1}^{2} \int_0^x \phi_n \phi_m dx = P_{mn} \frac{h_{x}^{k+1} - h_{x}^{k}}{\Delta t} \]

\[ P_{mn} = S_j \sum_{n=1}^{2} \sum_{m=1}^{2} \int_0^x \phi_n \phi_m dx \]

---

Transient FEM Problems

So our governing equation can be recast in matrix form as:

\[ A_{mn} h_{x}^{k+1} + P_{mn} \frac{h_{x}^{k+1} - h_{x}^{k}}{\Delta t} = 0 \]

Recall we already know what \( A_{mn} \) is

\( A_{mn} = \frac{T}{\Delta x} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \)

So now we need to solve for \( P_{mn} \):

\[ P_{mn} = S_j \sum_{n=1}^{2} \sum_{m=1}^{2} \int_0^x \phi_n \phi_m dx \]
Transient FEM Problems

Evaluate the matrix:

\[ P_{11} = S \int_0^{\Delta x} \left[ 1 - \frac{x}{\Delta x} \right]^2 dx = S \int_0^{\Delta x} 1 - 2 \frac{x}{\Delta x} + \frac{x^2}{\Delta x^2} dx \]

\[ P_{11} = S \int_0^{\Delta x} x - \frac{x^2}{\Delta x} + \frac{x^3}{3\Delta x^2} = S \frac{\Delta x}{3} \]

\[ P_{21} = P_{12} = S \int_0^{\Delta x} \left[ 1 - \frac{x}{\Delta x} \right] \frac{x}{\Delta x} dx = S \int_0^{\Delta x} \frac{x}{\Delta x} - \frac{x^2}{\Delta x^2} dx \]

\[ P_{21} = S \int_0^{\Delta x} \frac{x^2}{2\Delta x} - \frac{x^3}{3\Delta x^2} = S \frac{\Delta x}{6} \]

Transient FEM Problems

Evaluate the matrix:

\[ P_{22} = S \int_0^{\Delta x} \frac{x^2}{\Delta x^2} dx \]

\[ P_{nn} = S \frac{\Delta x}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \]

\[ P_{11} = S \int_0^{\Delta x} \frac{x^3}{3\Delta x^2} = S \frac{\Delta x}{3} \]

Recall:

\[ A_{\Delta x} = \begin{bmatrix} 1 & -1 \\ \Delta x & 1 \end{bmatrix} \]

\[ B_{\Delta x} = \begin{bmatrix} \Delta x \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \end{bmatrix} \]

Putting it all Together:

\[ S \frac{\Delta x}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} h^{k+1} - h^k + \frac{T}{\Delta x} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} h^{k+1} = 0 \]
Fully Implicit Formulation

\[
\left[ A_{nn} + \frac{P_{nn}}{\Delta t} \right] h_n^{k+1} = P_{nn} h_n^k
\]

Writing out the equations for element 1, nodes “1” and “2”:

\[
\begin{bmatrix}
A_{11} + P_{11}/\Delta t & A_{12} + P_{12}/\Delta t & 0 & h_1^{k+1} \\
A_{21} + P_{21}/\Delta t & A_{22} + P_{22}/\Delta t & 0 & h_2^{k+1} \\
0 & 0 & 0 & h_n^{k+1}
\end{bmatrix}
= \begin{bmatrix}
[P_{11}/\Delta t] h_1^k + [P_{12}/\Delta t] h_2^k \\
[P_{21}/\Delta t] h_1^k + [P_{22}/\Delta t] h_2^k \\
0 & 0
\end{bmatrix}
\]

\[\text{element - 1}\]