Darcy’s Law

• Last time
  – Groundwater flow is in response to gradients of mechanical energy
  – Three types
    • Potential
    • Kinetic
      – Kinetic energy is usually not important in groundwater
    • Elastic (compressional)
  – Fluid Potential, $\Phi$
    – Energy per unit mass
  – Hydraulic Head, $h$
    – Energy per unit weight
    – Composed of
      » Pressure head
      » Elevation head

• Today
  – Darcy’s Law
  – Hydraulic Conductivity
  – Specific Discharge
  – Seepage Velocity
    • Effective porosity

http://biosystems.okstate.edu/darcy/index.htm

Darcy’s Law

Henry Darcy, a French hydraulic engineer interested in purifying water supplies using sand filters, conducted experiments to determine the flow rate of water through the filters.

Published in 1856, his conclusions have served as the basis for all modern analysis of ground water flow

A FEW CAREER HIGHLIGHTS:

• In 1828, Darcy was assigned to a deep well drilling project that found water for the city of Dijon, in France, but could not provide an adequate supply for the town. However, under his own initiative, Henry set out to provide a clean, dependable water supply to the city from more conventional spring water sources. That effort eventually produced a system that delivered 8 m$^3$/min from the Rosoir Spring through 12.7 km of covered aqueduct.

• In 1848 he became Chief Director for Water and Pavements, Paris. In Paris he carried out significant research on the flow and friction losses in pipes, which forms the basis for the Darcy-Weisbach equation for pipe flow.

• He retired to Dijon and, in 1855 and 1856, he conducted the column experiments that established Darcy’s law for flow in sands.

**Darcy’s Law**

Cartoon of a Darcy experiment:

![Diagram of Darcy’s Law](image)

- **Datum**
- **Sand-filled column with bulk cross-sectional area $A$**
- **$Q$ - Rate of discharge [L$^3$/T]**
- **Constant head tanks at each end**

In this experiment, water moves due to gravity.

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**What is the relationship between discharge (flux) $Q$ and other variables?**

A plot of Darcy’s actual data:

![Plot of Darcy’s data](image)

http://biosystems.okstate.edu/darcy/index.htm
What is the relationship between discharge (flux) $Q$ and other variables?

Darcy’s Law

Darcy also found that if he used different kinds of sands in the column, discharge $Q$ changed, but for a particular sand, regardless of $Q$:
Why is there a minus sign in Darcy’s Law?
(Bradley and Smith, 1995)

Darcy’s Law

\[ Q = -KA \frac{dh}{dl} \]

Invert Darcy’s Law to express conductivity in terms of discharge, area, and gradient:

\[ K = \frac{Q}{A} \left[ -\frac{1}{\frac{dh}{dl}} \right] \]

This is how we measure conductivity:

*Imagine trying to measure gradient in a complex geology with three-dimensional flow and few observation points.*
Darcy’s Law

What are the dimensions of $K$? Dimensional analysis:

$$K = -\frac{Q \, dl}{A \, dh} = \left[ \frac{(L^3 T^{-1})(L)}{(L^2)(L)} \right] = \left[ \frac{L}{T} \right]$$

Darcy’s Law

$$Q = -KA \frac{dh}{dl}$$

This expresses Darcy’s Law in terms of discharge. We can also express it in terms of “Darcy velocity” or “specific discharge”, that is, discharge per unit bulk area, $A$. 
Darcy’s Law

We will look at three major topics important to Darcy’s Law:

- Hydraulic Head Gradient
- Bulk cross-sectional area of flow
- Hydraulic Conductivity (next time)

\[ Q = -KA \frac{dh}{dl} \]

Hydraulic Head

- Head is a measure of the total mechanical energy per unit weight.
- If \( K, Q \) and \( A \) don’t change with distance, then
Hydraulic Head

• In Darcy’s experiment, do the drops falling from the constant head tanks have constant velocity?

No!

Water droplets accelerate at 9.8 m/s².

Assuming no air resistance, the falling water drop has its potential energy converted to kinetic energy as it falls:

But water through our column has constant velocity—why?
Hydraulic Head

But water through our column has constant velocity—why?

In this experiment, with constant $K$, $Q$ and $A$, the head drops linearly with distance, and the specific discharge is constant.

$$q = -K \frac{dh}{dl} = \text{constant}$$

Darcy’s Law

- Could we use Darcy’s Law to model the falling drops?

- Darcy’s Law works because the driving forces (gravity and pressure) in the fluid are balanced by the viscous resistance of the medium.
Darcy’s Law

• What happens if the head gradient is too steep?
  – The fluid will have enough energy to accelerate in spite of the resistance of the grains, and inertial forces become important.
  – In this case potential energy (head) is not dissipated linearly with distance and Darcy’s Law does not apply.

• How can we tell when this occurs?

Reynolds Number

• How can we tell when this occurs?
• There are two standard, simple models used to explain Darcy’s Law, and thus to explore the Reynolds number:
  – Flow in a tube,
  – Flow around an object, usually a cylinder or sphere,
  • say, representing a grain of sand.

Flow in the vicinity of a sphere

Flow visualization (see slide 32)
Flow in the vicinity of a sphere

• Inertial force per unit volume at any location:
  \[ \rho \frac{du}{dx} \]
  \[ \rho = \text{density \left[ M/L^3 \right]} \]
  \[ u = \text{local fluid velocity \left[ L/T \right]} \]
  \[ U = \text{mean approach velocity \left[ L/T \right]} \]
  \[ \mu = \text{fluid dynamic viscosity \left[ M/LT \right]} \]
  \[ \nu = \text{fluid kinematic viscosity \left[ L^2/T \right]} \]
  \[ x, y = \text{Cartesian coordinates \left[ L \right]} \]
  \[ x \text{ in the direction of free stream velocity, } U \]

• Viscous force per unit volume:

• Dynamic similarity: \( Re = \)

  \[ Re = \text{constant for similitude} \]

Don’t worry about where the expressions for forces come from, H503 students, see Furbish, p. 126.

Flow in the vicinity of a sphere

Using a dimensional analysis approach, assume that functions vary with characteristic quantities \( U \) and \( R \), thus

\[ u \propto \]
\[ \frac{\partial u}{\partial x} \propto \]
\[ \frac{\partial^2 u}{\partial y^2} \propto \]

\[ \rho = \text{density \left[ M/L^3 \right]} \]
\[ u = \text{local fluid velocity \left[ L/T \right]} \]
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\[ x, y = \text{Cartesian coordinates \left[ L \right]} \]
\[ x \text{ in the direction of free stream velocity, } U \]
Flow in the vicinity of a sphere

• Inertial force per unit volume at any location:
  \[ \rho \frac{\partial u}{\partial x} \]

• Viscous force per unit volume:
  \[ \mu \frac{\partial^2 u}{\partial y^2} \]

• Dynamic similarity:

  \[ Re = \]

Reynolds Number

• For a fluid flow past a sphere
  \[ Re = \frac{\rho UR}{\mu} = \frac{UR}{\nu} \]

• For a flow in porous media?

  \[ Re = \frac{q}{\mu} \]

  – where characteristic velocity and length are:
  \[ q = \text{specific discharge} \]
  \[ L = \text{characteristic pore dimension} \]

  For sand \( L \) usually taken as the mean grain size, \( d_{50} \)
Reynolds Number

- When does Darcy’s Law apply in a porous media?
  - For $Re < 1$ to $10$,
  - For $Re > 1$ to $10$,

Let's revisit flow around a sphere to see why this non-linear flow happens.

$$Re = \frac{\rho q d_{50}}{\mu} = \frac{qd_{50}}{v}$$

Flow in the vicinity of a sphere

Very low Reynolds Number

mean flow

Laminar, linear flow

24. Circular cylinder at $Re=1.54$. At this Reynolds number the streamline pattern has clearly lost the fore-and-aft symmetry of figure 6. However, the flow has not yet separated at the rear. That begins at about $Re=5$.

$Re = \frac{\rho UR}{\mu} = \frac{UR}{v}$

though the value is not known accurately. Streamlines are made visible by aluminium powder in water. Photograph by Sudinski Taneda.

Source: Van Dyke, 1982, An Album of Fluid Motion
Flow in the vicinity of a sphere
Low Reynolds Number

41. Circular cylinder at $R=13.1$. The standing eddies become elongated in the flow direction as the speed increases. Their length is found to increase linearly with Reynolds number until the flow becomes unstable above $R=40$. (Taneda 1956)

“In these examples of a flow past a cylinder at low flow velocities a small zone of reversed flow - a separation bubble - is created on the lee side as shown (flow comes from the left).”

$L\text{aminar but non-linear flow. Non-linear because of the separation & eddies}$

$L$er $\mu$st $=U\frac{R}{v}$, The numerical values of $Re$ for cylinders, are not necessarily comparable to $Re$ for porous media.

Source: Van Dyke, 1982, An Album of Fluid Motion

Flow in the vicinity of a sphere
High Reynolds Number → transition to turbulence

47. Circular cylinder at $R=2000$. At this Reynolds number one may properly speak of a boundary layer. It is laminar over the front, separates, and breaks up into a turbulent wake. The separation points, moving forward as the Reynolds number is increased, have now attained their upstream limit, ahead of maximum thickness. Visualization is by air bubbles in water. ONERA photograph, Werlé & Guille 1972

$L$er $\mu$st $=U\frac{R}{v}$

Source: Van Dyke, 1982, An Album of Fluid Motion
Experimentally observed velocity variation (here in space)

**Velocity**
- temporal covariance
- spatial covariance

**Highly chaotic**

It is difficult to get turbulence to occur in a porous media. Velocity fluctuations are dissipated by viscous interaction with ubiquitous pore walls:

You may find it in flow through bowling ball size sediments, or near the bore of a pumping or injection well in gravel.
Reynolds Number

• When does Darcy’s Law apply in a porous media?
  – For $Re < 1$ to $10$,
    • flow is laminar and linear,
    • Darcy’s Law applies
  – For $Re > 1$ to $10$,
    • flow is still laminar but no longer linear
    • inertial forces becoming important
    – (e.g., flow separation & eddies)
    • linear Darcy’s Law no longer applies

$\textit{Specific Discharge, } q$

$Q = -KA \frac{dh}{d\ell}$

Suppose we want to know water “velocity.” Divide $Q$ by $A$ to get the volumetric flux density, or specific discharge, often called the Darcy velocity:

$\frac{Q}{A} = \frac{L^3 T^{-1}}{L^2} = \frac{L}{T}$ \rightarrow \text{units of velocity}$

$\frac{Q}{A} = q = -K \frac{dh}{d\ell}$

By definition, this is the discharge per unit bulk cross-sectional area.

But if we put dye in one end of our column and measure the time it takes for it to come out the other end, it is much faster suggested by the calculated $q$ – why?
Bulk Cross-section Area, \( A \)

\[ Q = -KA \frac{dh}{d\ell} \]

Effective Porosity, \( n_e \)

In some materials, some pores may be essentially isolated and unavailable for flow. This brings us to the concept of effective porosity.

\[ n_e = \]
Average or seepage velocity, \( v \).

- Actual fluid velocity varies throughout the pore space, due to the connectivity and geometric complexity of that space. This variable velocity can be characterized by its mean or average value.
- The average fluid velocity depends on
  - how much of the cross-sectional area \( A \) is made up of pores, and how the pore space is connected.
- The typical model for average velocity is:

\[
v = \text{spatial average of local velocity, } v'
\]

Darcy’s Law

- Review
  - Darcy’s Law
    - \( Re \) restrictions
  - Hydraulic Conductivity
  - Specific Discharge
  - Seepage Velocity
    - Effective porosity

- Next time
  - Hydraulic Conductivity
  - Porosity
  - Aquifer, Aquitard, etc