Transient Radial Flow Toward a Well

Aquifer Equation, based on assumptions becomes a 1D PDE for $h(r,t)$:

- transient flow in a homogeneous, isotropic aquifer
  \[ \nabla^2 h = \frac{S_s}{K} \frac{\partial h}{\partial t} \]

- fully penetrating pumping well & infinite, horizontal, confined aquifer of uniform thickness, thus essentially horizontal groundwater flow
  \[ \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \]

- flow symmetry: radially symmetric flow
  \[ \frac{1}{r} \frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} = \frac{S}{T} \frac{\partial h}{\partial t} \]

Boundary conditions:
- 2\textsuperscript{nd} order in space PDE for $h(r,t)$, need two BC's in space, at two $r$'s, say $r_1$ and $r_2$; in general we’ll pick $r_1=r_w, r_2=\infty$
- One Dirichlet BC at infinity and one Neuman at well radius, $r_w$, where we assume we know the pumping rate, $Q_w$.

Transient Radial Flow Toward a Well

During a pumping test, we typically measure drawdown (as opposed to head), so let’s set up the equation in terms of drawdown.

Assume initial potentiometric surface is horizontal everywhere.

\[ s = h_0 - h(t) \Rightarrow h = \]

\[ \frac{\partial h}{\partial r} = \text{,} \quad \frac{\partial h}{\partial t} = \text{.} \]

\[ \Rightarrow \quad \frac{1}{r} \frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial r^2} = \frac{S}{T} \frac{\partial s}{\partial t} \]
**Transient Flow Well Test Analysis**

To solve, we need boundary conditions and initial conditions

**I.C.:**

**B.C.s:** $2^{nd}$ order equation, we need two BCs

Use $r = 0, r = \infty$

Use continuity at the pumping well, treated as a line sink ($r = 0$)

At $r = 0$, $Q = -KA \frac{\partial h}{\partial r} = KA \frac{\partial s}{\partial r}$  \[ A = \]

$Q = 2\pi r b K \frac{\partial s}{\partial r} = $

\[\lim_{r \to \infty} \left( r \frac{\partial s}{\partial r} \right) \to \frac{Q}{2\pi T} \quad \text{(requires constant Q)}\]

$s \to \text{ as } r \to \infty \quad \text{(for all t)}$
**Transient Flow Well Test Analysis**

Solution:

\[ s(r, t) = \int_u^\infty \frac{e^{-u}}{u} du \]

where \( u = \frac{r^2 S}{4 T t} \)

In mathematics:

In hydrogeology:

\[ W(u) = \int_u^\infty \frac{e^{-u}}{u} du \]

**Theis Equation**

\[ u = \frac{r^2 S}{4 T t} \]

\[ W(u) = -\gamma - \ln u - \sum_{i=1}^{\infty} \frac{-u^i}{i \cdot i!} \]

\( \gamma = \) Euler’s constant = 0.577215

Values for \( W(u) \) for various values of \( u \) are also tabulated in numerous references.
Transient Flow Well Test Analysis

\[ s = \frac{Q}{4\pi T} W(u) \quad u = \frac{r^2 S}{4 T t} \]

Our goal in aquifer testing is to find \( T \) and \( S \); solving the drawdown equation shown above for \( T \) is problematic—\( T \) appears twice.

One way to solve for \( T \) is to compare a plot of \( s \) versus \( t \) to a “normal” graphic solution (when plotting data from numerous observation wells, we can typically normalize \( s \) by plotting it as a function of \( t/r^2 \) instead of \( t \))

Transient Flow Well Test Analysis

“Type curve method” for solving the Theis Equation

• Plot \( s \) versus \( t \) (or \( s \) versus \( t/r^2 \)) on log-log paper
• On another sheet of the same paper, plot \( W(u) \) versus \( 1/u \)
  Why \( 1/u \)? Our plot has \( t \) in the numerator, but in the equation defining \( u \), \( t \) appears in the denominator

\[ u = \frac{r^2 S}{4 T t} \]

• Match data (keep graph axes parallel)
• Pick “match point”—easiest to pick a match point with simple numbers—the match point does not have to fall on the curve.
• Solve equation
Transient Flow Well Test Analysis

(Schwartz and Zhang, 2003)

Q = 500 m$^3$/d
r = 300 m

(Schwartz and Zhang, 2003)
Transient Flow Well Test Analysis

\[ W(u) = \frac{1}{u_o} \]
\[ t = t_o \]
\[ s = s_o \]

\[ T = \frac{Q}{4\pi s_o} \]
\[ W(u_o) = \frac{500 \text{ m}^3}{4\pi (0.78 \text{ m})} \cdot 1 = \]

\[ S = \frac{4T t_o u_o}{r^2} = 4T u_o \left( \frac{t}{r^2} \right)_o = \frac{4 \left( \frac{51 \text{ m}^2}{\text{d}} \right) \left( \frac{22 \text{ min}}{1440 \text{ min}} \right) \left( \frac{1 \text{ d}}{300 \text{ m}} \right)}{0.1} = \]

But S is hard to measure—typically should only be reported to one significant digit \textit{id est} $3 \times 10^{-6}$

Transient Flow Well Test Analysis

Assumptions:

• Homogeneous, isotropic aquifer (we used a simplified form of the continuity equation instead of including the tensor for K)
• Constant water properties
• Infinite aquifer (used to develop boundary condition)
• Aquifer is confined; confining units are impermeable
• All flow is radial and horizontal
• Water is withdrawn instantaneously with decline in head
  (not typically the case for leakage from confining units or low k lenses)
• Well diameter is infinitesimal
  (water can be stored in the well bore, leading to "delayed" drawdown)
• Constant discharge from well
• C.E. Jacob (1965-1970)
  • Hantush’s mentor, well hydraulics, theory of leaky aquifers
  • 3 faculty, including Frank Titus and William Brutsaert
  • merged with Geology and Geophysics to form Geosciences Department
  • died of heart attack in 1970; Hantush returned briefly

Transient Radial Flow Toward a Well

Simplified approach to transient analysis: The Jacob Approximation

A simplifying assumption that makes solving the Theis equation easier

C. E. Jacob (1940) “Jacob approximation” “Jacob-Cooper equation”
  “Cooper!Jacob method” “Cooper-Jacob straight-line method”

\[ W(u) = -\gamma - \ln u - \sum_{i=1}^{\infty} \frac{-u^i}{i \cdot i!} \]

\[ W(u) = -0.577216 - \ln u + u - \frac{u^2}{2 \cdot 2!} + \frac{u^3}{3 \cdot 3!} - \frac{u^4}{4 \cdot 4!} + \ldots \]
**Transient Flow Well Test Analysis**

For $u \leq 0.01, \quad \sum_{i} u^{i} i^{i} = W(u)$

\[ u = \frac{r^{2}S}{4Tt} \]

So we need $t$ to be large for $u$ to be small

For $u \leq 0.01, s = \$

\[ s = \frac{Q}{4\pi T} \left[-\ln(1.781) - \ln\left(\frac{r^{2}S}{4Tt}\right)\right] \]

\[ s = \frac{Q}{4\pi T} \left[\ln(0.5615) + \ln\left(\frac{4Tt}{r^{2}S}\right)\right] \]

\[ s = \frac{Q}{4\pi T} \ln\left(\frac{0.5615}{r^{2}S}\right) \]
Transient Flow Well Test Analysis

\[ s = \frac{Q}{4 \pi T} \ln \left( \frac{(0.5615)4 T t}{r^2 S} \right) \]

Jacob-Cooper simplification

\[ \ln(u) = 2.3 \log u \]

Transient Flow Well Test Analysis

Suppose we measure drawdown at two times: \( t_1, t_2 \)

\[ s_2 - s_1 = \frac{Q}{4 \pi T} \left[ \ln\left( \frac{2.25 T}{r^2 S} \right) + \ln t_2 - \ln\left( \frac{2.25 T}{r^2 S} \right) - \ln t_1 \right] \]

\[ s_2 - s_1 = \]

\[ s_2 - s_1 = \]
Transient Flow Well Test Analysis

\[
s_2 - s_1 = \frac{Q}{4\pi T} \ln \frac{t_2}{t_1}
\]

Pick \( t_2 = 10 \, t_1 \); head change per log cycle = \( \Delta s_{lc} = s_2 - s_1 \)

\[
s_2 - s_1 = \frac{Q}{4\pi T} \ln \frac{10t_1}{t_1} = 2.3 \frac{Q}{4\pi T}
\]

Transient Flow Well Test Analysis

At \( t_0, s_\alpha = 0 = \frac{Q}{4\pi T} \ln \left( \frac{2.25 \, T \, t_\alpha}{r^2 S} \right) \)

\[
0 = \frac{Q}{4\pi T} \ln \left( \frac{2.25 \, T \, t_\alpha}{r^2 S} \right)
\]

\[
0 = \ln \left( \frac{2.25 \, T \, t_\alpha}{r^2 S} \right)
\]

\[
\Rightarrow e^0 = e^{\ln \left( \frac{2.25 \, T \, t_\alpha}{r^2 S} \right)}
\]
Transient Flow Well Test Analysis

\[ e^0 = e^{\ln\left(\frac{2.25 T_t o}{r^2 S}\right)} \]

\[ 1 = \frac{2.25 T_t o}{r^2 S} \]

(Schwartz and Zhang, 2003)

Q = 500 m³/d

Semi-log plot (drawdown versus log time)

(11 min, \( x_2 = 4.39 \) m)

(9 min, \( x_1 = 2.58 \) m)

(4.4 min, \( x_1 = 0 \))

(4.4 min, \( x_1 = 0 \))

(Schwartz and Zhang, 2003)
Transient Flow Well Test Analysis

\[ T = \frac{2.3 Q}{4\pi \Delta s_{lc}} \]

\[ T = \frac{2.3 \left( 500 \frac{m^3}{d} \right)}{4\pi (1.81 \text{ m})} = \]

\[ S = \frac{2.25 T t_o}{r^2} \]

\[ S = \frac{2.25 \left( 51 \frac{m^3}{d} \right) \left( 3.4 \text{ min} \frac{1 \text{ d}}{1440 \text{ min}} \right)}{(300 \text{ m})^2} = \]

**Transient Flow Well Test Analysis**

Remember that the Jacob-Cooper method is predicated on the value of \( u \) being less than about 0.01!

\[ u = \frac{r^2 S}{4 T t} \]

\[ u = \frac{(300 \text{ m})^2 \left( 3 \times 10^{-6} \right)}{4 \left( 51 \frac{m^3}{d} \right) \left( 100 \text{ min} \frac{1 \text{ d}}{1440 \text{ min}} \right)} = 0.02 \]
**Transient Flow Well Test Analysis**

Instead of looking at drawdown in one well with time, we can also look at drawdown measured at the same time in two or more wells: “Distance-drawdown method”

\[
\begin{align*}
\text{s}_1 - \text{s}_2 &= \frac{Q}{4\pi T} \left[ \ln \left( \frac{2.25 \, T \, t}{r_1^2 \, S} \right) - \ln \left( \frac{2.25 \, T \, t}{r_2^2 \, S} \right) \right] \\
&= \ln \left( \frac{2.25 \, T \, t}{S} \right) - \ln \left( \frac{2.25 \, T \, t}{S} \right) + \ln r_2^2 \\
\end{align*}
\]

(Tran, 1984)
Transient Flow Well Test Analysis

\[ s_1 - s_2 = \frac{2Q}{4\pi T} \ln \frac{r_2}{r_1} \]

\[ s_1 - s_2 = \frac{Q \left( \ln \frac{r_2}{r_1} \right)}{2\pi T} \]

Distance-drawdown formula

To calculate S, we use a formula similar to that used in the time-drawdown formula.

\[ S = \frac{2.25 T t_o}{r^2} \]

We used \( t_o \) because it told us the time at which drawdown was zero; for a distance-drawdown formula, we want to know the distance at which \( s = 0 \) (we'll call it \( r_o \))—the rest of the derivation is the same as for the distance-drawdown method, so we'll jump right to the formula:

\[ S = \frac{2.25 T t}{r^2} \]

Distance-drawdown formula

Remember that there are two formulae for applying the Jacob-Cooper method—time-drawdown and distance-drawdown
**Transient Flow Well Test Analysis**

\[
Q = 220 \text{ gpm} = 29.41 \text{ ft}^3/\text{min} = 42,350 \text{ ft}^3/\text{d}
\]

Drawdowns measured at \( t = 220 \text{ min} \)

\[
T = \frac{2.3 Q}{4 \pi \Delta S_{lc}}
\]

\[
T = \frac{2.3 \left(42350 \frac{\text{ft}^3}{\text{d}}\right)}{4 \pi \left(15.7 \text{ ft}\right)}
\]

\[
S = \frac{2.25 T t}{r_o^2}
\]

\[
S = \frac{2.25 \left(987 \frac{\text{ft}^2}{\text{d}}\right) \left(220 \text{min}\right)}{(1900 \text{ ft})^2}
\]

\[
u = \frac{r^2 S}{4 T t}
\]

\[
u = \frac{\left(1900 \text{ ft}\right)^2 \left(9 \times 10^{-5}\right)}{4 \left(987 \frac{\text{ft}^2}{\text{d}}\right) \left[(220 \text{min})\left(\frac{1 \text{d}}{1440 \text{min}}\right)\right]}
\]

(Schwartz and Zhang, 2003)
Transient Flow Well Test Analysis

\[ T = \frac{Q \left( \ln \frac{r_2}{r_1} \right)}{2 \pi \Delta s} \]

Distance-drawdown formula  Thiem equation

Formulation of the DDF is identical to that of the Thiem equation

This means that when Jacob’s approximation applies, the difference in drawdown between any two points stabilizes—the potentiometric surface is being drawn down uniformly with time.
**Transient Flow Well Test Analysis**

Why doesn't this apply in early time? Stability is achieved when the slope of the potentiometric surface has the correct gradient to supply Q to the well.

We are constantly lowering the potentiometric surface—we must do this to release water from storage. In early time, water is only released from storage near the well, but as time gets large, water is supplied from a larger area.

As the volume of aquifer supplying Q increases, less change in head is needed to yield the same amount of water, so the cone of depression increases at a decreasing rate.
How do cones of depression vary from “normal” with changes in aquifer properties, and how does this affect the drawdown hydrograph of a monitoring well?

How might drawdown hydrographs vary from “normal” in situations where our pumping test assumptions are not met?

Variations in drawdown
Cone of depression as a function of time

\[
T = 1000 \text{ gpd/ft; } S = 10^{-4}; \quad Q = 300 \text{ gpm}
\]

(Hall, 1996)
Variations in drawdown
Variations with pumping rate

Q = 100 gpm
Q = 300 gpm
Q = 600 gpm

T = 1000 gpd/ft;
S = 10^{-4};
r = 100 ft

Variations in drawdown
Variations with storativity

S = 0.000001
S = 0.0001
S = 0.1
S = 10^{-4};
Q = 300 gpm;
r = 100 ft

(Hall, 1996)
Variations in drawdown
Variations with transmissivity

\[ T = 500 \text{ gpd/ft} \quad T = 1,000 \text{ gpd/ft} \quad T = 2,500 \text{ gpd/ft} \]

S = 1000 gpd/ft; 
Q = 300 gpm; 
r = 100 ft

Sensitivity to T and S summarized

Low S:
Cone shape remains similar, the cone is just bigger

High S:

Low T: tight, deep cone;    High T: wide, shallow cone

Low S, creates greater drawdown to produce a given volume of water; cone of depression shape stays the same

Low T: hard to transmit water, most efficient production is from close in, creates steeper gradients of s to produce water from close in.