The Seismic Wave Equation

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Waves in one dimension. The wave equation is a partial differential equation that relates second time and spatial derivatives of propagating wave disturbances in a simple way. For a *nondispersive* system (where all frequencies of excitation propagate at the same velocity), the formula for sinusoidal or *harmonic* waves of displacement with amplitude A as a function of space and time is just

$$u_y(x,\ t) = A\sin(kx - \omega t) = A\sin(kx - kct) \tag{1}$$

where $k = 2\pi/\lambda = \omega/c$ is the *wavenumber* for a disturbance of wavelength λ, $\omega = 2\pi f$ is the *radian frequency*, and c is the *phase velocity.* We can construct the 1-dimensional wave equation by noting that

$$\frac{\partial^2 u_y}{\partial x^2} = -k^2 A\sin(kx - kct) \tag{2}$$

and

$$\frac{\partial^2 u_y}{\partial t^2} = -k^2 c^2 A\sin(kx - kct) \tag{3}$$

Thus, the proportionality constant between the two second partial derivatives is just c^2, so that

$$\frac{\partial^2 u_y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u_y}{\partial t^2} \tag{4}$$

which is the *1-dimensional scalar wave equation.*

An instructional 1-dimensional wave system that we will examing before considering (the considerably more complicated) 3-d seismic wave system is transverse waves on a string aligned in the $\hat{x}$ direction, with a linear density ρ, and under a tension, ζ (e.g., a guitar string). To derive the properties of waves in this system (Figure 1) we apply the equation of motion, $\mathbf{F} = m\mathbf{a}$. For a string element displaced in the $\hat{y}$ direction, the net vertical force is

$$f_y = \zeta\sin\theta_2 - \zeta\sin\theta_1 \tag{5}$$

If the angles are small (this is the string equivalent of small strain), using

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \approx \theta \tag{6}$$

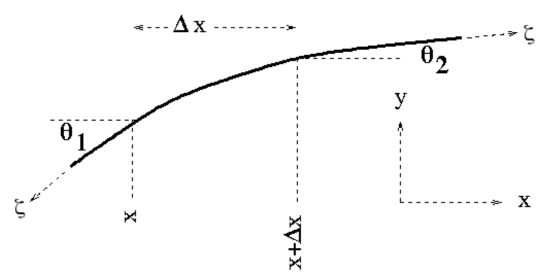

Figure 1: A Tensioned String Element in Disequilibrium

and

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \cdots \approx \theta \tag{7}$$

we have

$$\sin\theta \approx \tan\theta \approx \frac{\partial u_y}{\partial x} \tag{8}$$

and thus

$$f_y \approx \zeta \left(\left.\frac{\partial u_y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u_y}{\partial x}\right|_{x} \right) \equiv \zeta \Delta\xi \tag{9}$$

where

$$\Delta\xi = \left.\frac{\partial u_y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u_y}{\partial x}\right|_{x} \tag{10}$$

is the change in slope over the element. The equation of motion can now be written as

$$f_y = ma = \rho\Delta x \frac{\partial^2 u_y}{\partial t^2} = \zeta\Delta\xi \tag{11}$$

or

$$\rho \frac{\partial^2 u_y}{\partial t^2} = \zeta \frac{\Delta\xi}{\Delta x} \tag{12}$$

Taking the limit as the segment becomes small we have

$$\lim_{\Delta x \to 0} \frac{\Delta\xi}{\Delta x} = \frac{\partial^2 u_y}{\partial x^2} \tag{13}$$

so the equation of motion becomes

$$\frac{\partial^2 u_y}{\partial t^2} = \frac{\zeta}{\rho} \frac{\partial^2 u_y}{\partial x^2} \ . \tag{14}$$

(14) is just a 1-dimensional scalar wave equation for waves with a phase propagation velocity in the $\hat{x}$ direction of

$$c = \sqrt{\frac{\zeta}{\rho}} \ . \tag{15}$$

Note that the simplicity of this result hinges on the small angle approximation (8). This provides the necessary linear relationship between "stress" (the $\hat{y}$ component of tension) and "strain" ($\hat{y}$ displacement). A similar situation will occur with seismic waves, which commonly have small or *infinitesimal* strains in earthquake or exploration seismology, so that first order terms in the stress-strain relationship are adequate to characterize the constitutive relationship. Because of the linear stress-strain relationship, transverse waves on a string obey the principals of *superposition*, i.e., if $y_1(x,\ t)$ is a solution, and $y_2(x,\ t)$ is a solution, then so is $y_1 + y_2$. On the other hand, if the angles of the string perturbation become large, the first-order stress-strain theory will not be an adequate representation of the balance of forces and displacements, and

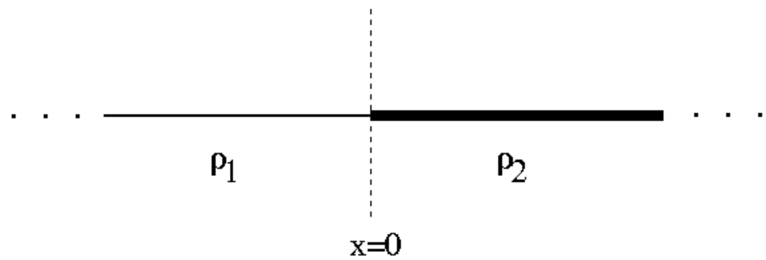

Figure 2: An Inhomogeneous String

the system will become appreciably *nonlinear*. Nonlinear systems do not obey superposition, and, also unlike linear systems, their behavior also depends on the amplitude of the disturbance.

Superposition makes the analysis of string, seismic, or other linear wave systems much easier. In particular, it enables us to apply Fourier theory. For a linear system, can decompose any propagating wave, $g(x,t)$ into its constituent harmonic components

$$g(x,t) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} \Phi(f,k) e^{\imath 2\pi f t} e^{-\imath 2\pi k x} df \ dk \tag{16}$$

(16) is an *inverse Fourier transform* and Φ is the $f-k$, or *frequency-wavenumber* spectrum of the wave $g(\omega t - kx)$. Note that k in this context is just $1/\lambda$ rather than the customary $2\pi/\lambda$ to make it analagous to $f = 1/T$, where T is the period. For a particular frequency, f_0, and wavenumber constituent of g, k_0, the complex Φ gives the amplitude and phase. To find Φ, we apply a *forward Fourier transform*

$$\Phi(f,k) = \int_{\infty}^{\infty} \int_{\infty}^{\infty} g(x,t) e^{-\imath 2\pi f t} e^{\imath 2\pi k x} dt \ dx \ . \tag{17}$$

For harmonic waves, which have only a single frequency, we can readily see what the f-k spectrum is by noting that

$$g(x, \ t) = A\cos(2\pi f_0 t - k_0 x) = \frac{A}{2}\left(e^{\imath(2\pi f_0 t - k_0 x)} + e^{-\imath(2\pi f_0 t - k_0 x)}\right) \ . \tag{18}$$

(16) tells us that the proper Φ to recover this function must be

$$\Phi(f,k) = \frac{A}{2}(\delta(f - f_0, k - k_0) + \delta(f + f_0, k + k_0)) \tag{19}$$

where $\delta(f,k)$ is the 2-dimensional delta function.

We next address the question as to how waves behave when there is a spatial change in properties (either density or elasticity). Consider an infinite-length spliced string where the density takes an abrupt change at $x = 0$ as x increases (Figure 2). The system is excited by a harmonic wave traveling to the right at $x = -\infty$. To investigate the wave-propagation physics of this simple inhomogeneous system, we construct a composite solution to the wave equation and apply physical matching conditions at the splice. Where $\rho = \rho_1$, we will have a general wave consisting of right-going (incident) and left-going (reflected) components will be

$$u_1(x,t) = Ae^{\imath(\omega t - k_1 x)} + Be^{\imath(\omega t + k_1 x)} \tag{20}$$

where

$$k_1 = \frac{\omega}{c_1} = \omega\sqrt{\frac{\rho_1}{\zeta}}\ . \tag{21}$$

and where $\rho = \rho_2$, we will have only a right-going (transmitted) component

$$u_2(x,\ t) = Ce^{\imath(\omega t - k_2 x)} \tag{22}$$

where

$$k_2 = \frac{\omega}{c_2} = \omega\sqrt{\frac{\rho_2}{\zeta}}\ . \tag{23}$$

Our first matching condition has already been applied; as we are exciting the splice by a harmonic disturbance of frequency ω, all other wave components in the system will also only consist of harmonic disturbances of the same frequency. This is a consequence of linear systems; it would not generally be true in a nonlinear system. The second matching condition is that the string does not disassociate, so that the $\hat{y}$ displacement will be continuous at the splice

$$u_1(0,t) = u_2(0,t) \tag{24}$$

which implies that

$$A + B = C\ . \tag{25}$$

The final matching condition is that the $\hat{y}$ forces are continuous at the splice. This is analogous to the continuity of stress that we discussed earlier for elastic media (a force discontinuity within a connected string would result in a string element having an infinite acceleration). The force balance is

$$f_{0-} = f_{0+} = \zeta\ \frac{\partial u_1}{\partial x}\bigg|_{x=0^-} = \zeta\ \frac{\partial u_2}{\partial x}\bigg|_{x=0^+} \tag{26}$$

which implies that

$$-\zeta k_1 A + \zeta k_1 B = -\zeta k_2 C \tag{27}$$

or

$$\zeta k_1(A - B) = \zeta k_2 C\ . \tag{28}$$

Converting the wavenumbers to frequency, densities, and tension gives

$$\zeta(A-B)\omega\sqrt{\frac{\rho_1}{\zeta}} = \zeta C\omega\sqrt{\frac{\rho_2}{\zeta}} = (A-B)\rho_1\sqrt{\frac{\zeta}{\rho_1}} = (A-B)\rho_1 c_1 = C\rho_2 c_2 \tag{29}$$

where the density-velocity products are called *acoustic impedances*.

To solve for the *reflection and transmission coefficients*, $R_{12} = B/A$ and $T_{12} = C/A$, respectively we need to solve the linear system of equations

$$\begin{pmatrix} 1 & -1 \\ 1 & \frac{\rho_2 c_2}{\rho_1 c_1} \end{pmatrix} \begin{pmatrix} R_{12} \\ T_{12} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} . \tag{30}$$

(30) is easily solved to obtain the coefficients

$$R_{12} = \frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = \frac{B}{A} \tag{31}$$

and

$$T_{12} = \frac{2\rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = \frac{C}{A} . \tag{32}$$

Note that the coefficients depend only on the density change, which should be expected as the only other physical property, the tension, is constant in this system. Note that if $\rho_1 < \rho_2$, then $T < 1$, and the amplitude of the transmitted wave will be less than that of the incident wave, while if $\rho_2 < \rho_1$, then $T > 1$, and the amplitude of the transmitted wave is greater than that of the incident wave.

Reversing the indices, gives the equivalent expressions for waves incident from the right

$$R_{21} = \frac{\rho_2 c_2 - \rho_1 c_1}{\rho_1 c_1 + \rho_2 c_2} = -R_{12} \tag{33}$$

and

$$T_{21} = \frac{2\rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2} = 2 - T_{12} . \tag{34}$$

We can evaluate the kinetic and potential energy in waves on a string by noting that the kinetic energy of an element of length dx is

$$E_K = \frac{1}{2} m v^2 = \frac{1}{2} \rho \left(\frac{\partial u_y}{\partial t} \right)^2 \cdot dx \tag{35}$$

and its potential energy stored as strain is

$$E_P = \zeta \cdot \epsilon \cdot dx \tag{36}$$

where ϵ is the strain, which in this case is just the normalized lengthening of the string element

$$\epsilon = \frac{\sqrt{(dx^2) + (du_y)^2} - dx}{dx} = \sqrt{1 + \left(\frac{\partial u_y}{\partial x} \right)^2} - 1 . \tag{37}$$

For small slopes, this gives

$$\epsilon = 1 + \frac{1}{2} \left(\frac{\partial u_y}{\partial x} \right)^2 - 1 = \frac{1}{2} \left(\frac{\partial u_y}{\partial x} \right)^2 \tag{38}$$

so that for infinitesimal strains we have

$$E_P = \frac{1}{2}\zeta\left(\frac{\partial u_y}{\partial x}\right)^2 dx \ . \tag{39}$$

Now consider a harmonic wave

$$u(x,\ t) = A\cos(\omega t - kx) \ . \tag{40}$$

The potential energy density (energy/length) averaged over one wavelength is

$$E_{P\lambda} = \frac{A^2}{2\lambda}\int_0^{\lambda} \zeta k^2\ \sin^2(\omega t - kx)\ dx \ = \ \frac{A^2\zeta k^2}{2\lambda}\int_0^{\lambda} \sin^2(\omega t - kx)\ dx \ . \tag{41}$$

Substituting $\lambda = 2\pi/k$ and $\theta = \omega t - kx$ gives

$$E_{P\lambda} = \frac{A^2\zeta k^3}{4\pi}\int_0^{2\pi/k} \sin^2(\omega t - kx)\ dx = \frac{-A^2\zeta k^2}{4\pi}\int_{\omega t}^{\omega t - 2\pi} \sin^2\theta\ d\theta \tag{42}$$

$$= \frac{A^2\zeta k^2}{4\pi}\int_0^{2\pi} \sin^2\theta\ d\theta = \frac{A^2\zeta k^2}{4} = \frac{A^2\rho c^2 k^2}{4} = \frac{A^2\rho\omega^2}{4} \ .$$

Following a similar development for the energy density over one wavelength, we have

$$E_{K\lambda} = \frac{A^2\rho}{2\lambda}\int_0^{\lambda} \omega^2\ \sin^2(\omega t - kx)\ dx \tag{43}$$

$$= \frac{A^2 k\rho}{4\pi}\int_0^{2\pi/k} \omega^2\ \sin^2(\omega t - kx)\ dx \ = \ \frac{A^2\rho\omega^2}{4} = E_{P\lambda} \ .$$

Strain and kinetic energy in elastic waves are thus equal and are each proportional to density, and proportional to the square of amplitude and frequency. The total energy per wavelength is

$$E_{T\lambda} = E_{P\lambda} + E_{K\lambda} = \frac{A^2\rho\omega^2}{2} \tag{44}$$

Although all motion in string waves is perpendicular to the direction of propagation, the traveling wave still carries energy. The energy flux is given by

$$\dot{E} = E_T c = \frac{A^2\rho\omega^2 c}{2} \ . \tag{45}$$

The energy fluxes for incident, reflected, and transmitted waves for the inhomogeneous string are thus

$$\dot{E}_{incident} = \frac{A^2\rho_1\omega^2 c_1}{2} \tag{46}$$

$$\dot{E}_{refl} = \frac{A^2 R_{12}^2\rho_1\omega^2 c_1}{2} \tag{47}$$

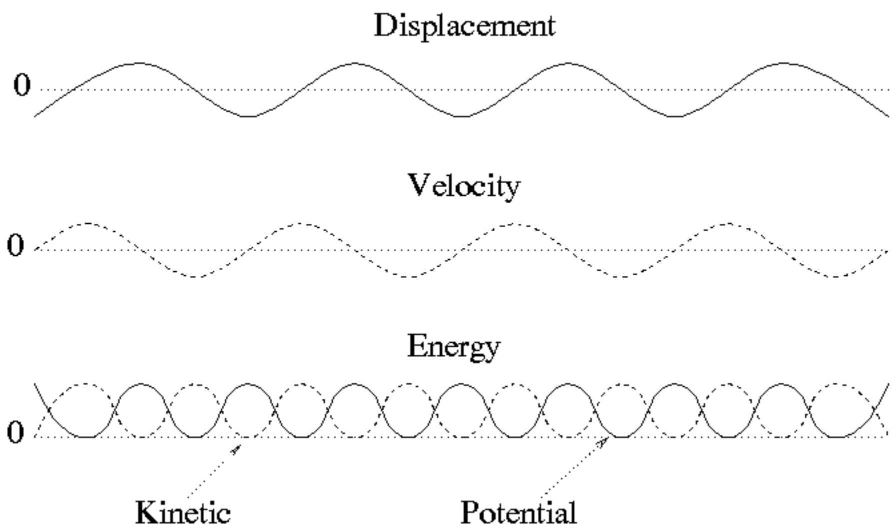

Figure 3: Harmonic Elastic Wave Displacement, Velocity, and Energy

$$\dot{E}_{trans} = \frac{A^2 T_{12}^2 \rho_2 \omega^2 c_2}{2} \ . \tag{48}$$

The total energy flux for waves generated at the splice is thus

$$\dot{E}_{refl} + \dot{E}_{trans} \tag{49}$$

$$= A^2 \left(\frac{(\rho_1 c_1 - \rho_2 c_2)^2}{(\rho_1 c_1 + \rho_2 c_2)^2} \cdot \frac{\rho_1 \omega^2 c_1}{2} + \frac{(2\rho_1 c_1)^2}{(\rho_1 c_1 + \rho_2 c_2)^2} \cdot \frac{\rho_2 \omega^2 c_2}{2} \right)$$

$$= \frac{A^2 \omega^2}{2} \left(\frac{4\rho_1^2 c_1^2 \rho_2 c_2 + (\rho_1 c_1 - \rho_2 c_2)^2 \cdot \rho_1 c_1}{(\rho_1 c_1 + \rho_2 c_2)^2} \right)$$

$$= \frac{A^2 \omega^2 \rho_1 c_1}{2} \left(\frac{(\rho_1 c_1)^2 + 2\rho_1 \rho_2 c_1 c_2 + (\rho_2 c_2)^2}{(\rho_1 c_1 + \rho_2 c_2)^2} \right) = \frac{A^2 \omega^2 \rho_1 c_1}{2} = \dot{E}_{incident}$$

so we have, as expected, energy flux conservation for the system.

Seismic waves. So far we have discussed some aspects of the static elastic behavior of elastic media, but have not dealt with dynamic properties. We will now return to applying the equation of motion

$$\mathbf{f} = \int_S \mathbf{f}_s dS + \int_V \mathbf{f}_b dV = m\mathbf{a} = m \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{50}$$

for elastic media subject to surface and body forces, $\mathbf{f_s}$ and $\mathbf{f_b}$, respectively.

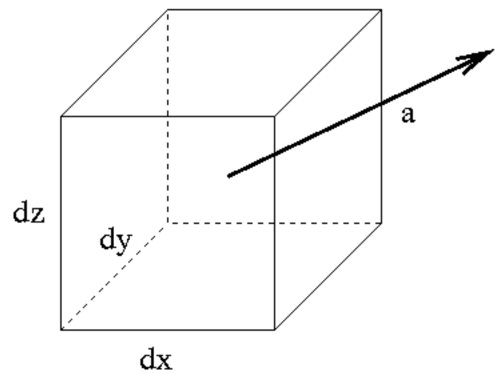

Figure 4: An Accelerated Element

Consider the total force acting on a small volume element of a connected medium with density ρ and volume $dV = dx\ dy\ dz$, which will be some combination of contact and body forces. The contact force from the $\hat{x}$ and $-\hat{x}$ faces is the sum of the tractions times their respective face areas ($dS = dy \cdot dz$ in both cases)

$$\int_{Sx} \mathbf{f}_{sx}\ dSx = (\tau_{x+} + \tau_{x-})\ dy\ dz =$$
$$\left(\left(\boldsymbol{\sigma} + \frac{\partial \boldsymbol{\sigma}}{\partial x}\ dx\right)\hat{x} + \boldsymbol{\sigma}(-\hat{x})\right)\ dy\ dz = \frac{\partial \boldsymbol{\sigma}}{\partial x}\hat{x} dV \tag{51}$$

where we have assumed that first derivatives are sufficient to propagate stress across the volume (this is o.k. because we know that tractions are continuous in a connected medium).

The body force term is

$$\int_V \mathbf{f}_{bx}\ dV = \mathbf{f}_{bx} dx\ dy\ dz\ . \tag{52}$$

The terms are similar for the other face pairs, giving a total force expression of

$$\left(\frac{\partial \sigma_{ij}}{\partial x_j} + f_{bi}\right) dV = m\frac{\partial^2 u_i}{\partial t^2} = \rho\frac{\partial^2 u_i}{\partial t^2} dV\ . \tag{53}$$

Dividing through by dV gives the *equation of motion for continuous media*

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_{bi} = \frac{m}{dV}\frac{\partial^2 u_i}{\partial t^2} = \rho\frac{\partial^2 u_i}{\partial t^2} \equiv \sigma_{ij,j} + f_{bi} \tag{54}$$

where the comma notation, $\sigma_{ij,j}$, indicates differentiation with respect to x_j, and repeated indices are summed in accordance with the Einstein summation convention. This equation must be satisfied for all points in a continuous medium.

Recall, for an isotropic medium, we have the constitutive relationship between stress and strain

$$\sigma_{ij} = \lambda \Theta \delta_{ij} + 2\mu \epsilon_{ij} \ . \tag{55}$$

Because

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{56}$$

the stresses on the $\hat{x}$ face are just

$$\sigma_{xx} = \lambda \Theta + 2\mu \frac{\partial u_x}{\partial x} \tag{57}$$

$$\sigma_{xy} = \mu \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \tag{58}$$

$$\sigma_{xz} = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \ . \tag{59}$$

The equation of motion (54) has terms which are the spatial derivatives of the stress, which are now seen to be

$$\frac{\partial \sigma_{xx}}{\partial x} = \lambda \frac{\partial \Theta}{\partial x} + 2\mu \frac{\partial^2 u_x}{\partial x^2} \tag{60}$$

$$\frac{\partial \sigma_{xy}}{\partial y} = \mu \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) \tag{61}$$

$$\frac{\partial \sigma_{xz}}{\partial z} = \mu \left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right) \ . \tag{62}$$

If there are no body forces, and the medium is homogeneous and isotropic, we thus have, for the $\hat{x}$ component of the equation of motion,

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \lambda \frac{\partial \Theta}{\partial x} + 2\mu \frac{\partial^2 u_x}{\partial x^2} \tag{63}$$

$$+\mu \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial x \partial y}\right) + \mu \left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial x \partial z}\right) \ .$$

Because

$$\frac{\partial \Theta}{\partial x} = \frac{\partial}{\partial x} \bigtriangledown \cdot \mathbf{u} = \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y \partial x} + \frac{\partial^2 u_z}{\partial z \partial x} \tag{64}$$

we can group terms to obtain

$$\rho \frac{\partial^2 u_x}{\partial t^2} = (\lambda + \mu) \frac{\partial \Theta}{\partial x} + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) \tag{65}$$

$$= (\lambda + \mu) \frac{\partial \Theta}{\partial x} + \mu \bigtriangledown^2 u_x \ .$$

The development is identical for the $\hat{y}$ and $\hat{z}$ components, so the general equation of motion in three dimensions for isotropic media is

$$\rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial \Theta}{\partial x_i} + \mu \bigtriangledown^2 u_i \tag{66}$$

or

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu) \bigtriangledown (\bigtriangledown \cdot \mathbf{u}) + \mu \bigtriangledown^2 \mathbf{u} \tag{67}$$

Using a vector calculus identity for the vector Laplacian

$$\bigtriangledown^2 \mathbf{u} = (\bigtriangledown^2 u_x, \;\; \bigtriangledown^2 u_y, \;\; \bigtriangledown^2 u_z) = \bigtriangledown(\bigtriangledown \cdot \mathbf{u}) - \bigtriangledown \times (\bigtriangledown \times \mathbf{u}) \tag{68}$$

gives

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu) \bigtriangledown (\bigtriangledown \cdot \mathbf{u}) - \mu \bigtriangledown \times (\bigtriangledown \times \mathbf{u}) \; . \tag{69}$$

To make sense of this complicated looking expression, we make use of a clever decomposition (Helmholtz' Theorem) whereby we write the displacement field as a sum of the gradient of a scalar displacement potential and the curl of a vector displacement potential

$$\mathbf{u}(\mathbf{x}, \; t) = \bigtriangledown \Phi(\mathbf{x}, \; t) + \bigtriangledown \times \mathbf{\Psi}(\mathbf{x}, \; t) \tag{70}$$

The physical reason for applying this decomposition is to separate the displacement field into two parts. One part will have zero divergence, as

$$\bigtriangledown \cdot (\bigtriangledown \times \mathbf{\Psi}) = 0 \tag{71}$$

for any vector field, $\mathbf{\Psi}$, and the other part will have zero curl, as

$$\bigtriangledown \times (\bigtriangledown \Phi) = \mathbf{0} \tag{72}$$

for any scalar field, Φ.

Substituting the Helmholtz potentials into the equation of motion gives

$$\begin{aligned} \rho \frac{\partial^2 (\bigtriangledown \Phi + \bigtriangledown \times \mathbf{\Psi})}{\partial t^2} &= (\lambda+2\mu) \bigtriangledown (\bigtriangledown \cdot (\bigtriangledown \Phi + \bigtriangledown \times \mathbf{\Psi})) - \mu \bigtriangledown \times (\bigtriangledown \times (\bigtriangledown \Phi + \bigtriangledown \times \mathbf{\Psi})) \\ &= (\lambda + 2\mu) \bigtriangledown (\bigtriangledown \cdot \bigtriangledown \Phi) - \mu \bigtriangledown \times (\bigtriangledown \times \bigtriangledown \times \mathbf{\Psi}) \; . \end{aligned} \tag{73}$$

We can simplify this somewhat by noting that

$$- \bigtriangledown \times (\bigtriangledown \times \bigtriangledown \times \mathbf{\Psi}) = - \bigtriangledown (\bigtriangledown \cdot (\bigtriangledown \times \mathbf{\Psi})) + \bigtriangledown^2 (\bigtriangledown \times \mathbf{\Psi}) = \bigtriangledown^2 (\bigtriangledown \times \mathbf{\Psi}) \tag{74}$$

and

$$\bigtriangledown \cdot (\bigtriangledown \Phi) = \bigtriangledown^2 \Phi \tag{75}$$

so that

$$\rho \frac{\partial^2 (\bigtriangledown \Phi + \bigtriangledown \times \mathbf{\Psi})}{\partial t^2} = (\lambda + 2\mu) \bigtriangledown (\bigtriangledown^2 \Phi) + \mu \bigtriangledown^2 (\bigtriangledown \times \mathbf{\Psi}) \; . \tag{76}$$

We next group the terms and pull out the gradient and curl operators to obtain

$$\bigtriangledown\left((\lambda+2\mu)\bigtriangledown^2\Phi-\rho\frac{\partial^2\Phi}{\partial t^2}\right)=\bigtriangledown\times\left(\mu\bigtriangledown^2\mathbf{\Psi}-\rho\frac{\partial^2\mathbf{\Psi}}{\partial t^2}\right)\ . \tag{77}$$

The left-hand side is the gradient of a function of Φ, while the right-hand side is the curl of a function of Ψ. As these sides are always equal for all t and $\mathbf{x}$, they must be equal to some constant, which we can take as zero. Thus, the Helmholtz formalization has enabled us (with much malice aforethought) to separate the elastodynamic equation of motion for an isotropic medium into two differential equations

$$\bigtriangledown^2\Phi(\mathbf{x},\ t)=\frac{\rho}{\lambda+2\mu}\frac{\partial^2\Phi(\mathbf{x},\ t)}{\partial t^2} \tag{78}$$

and

$$\bigtriangledown^2\mathbf{\Psi}(\mathbf{x},\ t)=\frac{\rho}{\mu}\frac{\partial^2\mathbf{\Psi}(\mathbf{x},\ t)}{\partial t^2}\ . \tag{79}$$

The first of these two differential equations in 3-space and time is a *scalar wave equation*, the second is a *vector wave equation.* They describe the two types of *seismic body waves* in isotropic, homogeneous elastic media.

Recall that the basic form of a wave equation in one dimension is

$$\frac{\partial^2u(x,\ t)}{\partial^2x}=\frac{1}{c^2}\frac{\partial^2u(x,\ t)}{\partial^2t} \tag{80}$$

where c is the phase velocity. Likewise, we can see that the two types of waves which we have derived for solid media have phase velocities of

$$\alpha=\sqrt{\frac{\lambda+2\mu}{\rho}} \tag{81}$$

and

$$\beta=\sqrt{\frac{\mu}{\rho}} \tag{82}$$

respectively. Note that these velocities are proportional to the square root of the moduli, and inversely proportional to the square root of the density. Phase velocities in crustal rocks are typical on the order of 2-6 km/s for P waves. For a typical Poisson solid rock with $\lambda=\mu=3\times10^{10}$ Pa, and density $\rho=3000$ kg/m^2, we have $\alpha\approx4.47$ km/s and $\beta=\alpha/\sqrt{3}\approx2.58$ km/s. At 1 Hz, the corresponding wavelengths are $\lambda_\alpha=4.47$ km and $\lambda_\beta=2.58$ km.

Recall, in the 1-dimensional wave equation, a harmonic plane wave solution is simply

$$u(x,\ t)=Ae^{\imath(\omega t\pm kx)} \tag{83}$$

where $k=2\pi/\lambda$ is the wavenumber, and $\omega=2\pi f$ is the radian frequency.

For plane waves in three spatial dimensions, we can write displacement harmonic solutions to the wave equation as

$$\mathbf{u}(\mathbf{x},\ t) = Ae^{\imath(\omega t \pm \mathbf{k}\cdot\mathbf{x})} \tag{84}$$

where the wavenumber object is now a wavevector specifying the direction of propagation

$$\mathbf{k} = (k_x,\ k_y, k_z) \tag{85}$$

(propagation is in the $+\hat{k} = \mathbf{k}/|\mathbf{k}|$ direction if the negative sign is used and in the $-\hat{k}$ direction if the positive sign is used) and a length proportional to the reciprocal of the wavelength

$$|\mathbf{k}| = \frac{2\pi}{\lambda}\ , \tag{86}$$

and

$$\mathbf{x} = (x,\ y,\ z)\ . \tag{87}$$

The phase velocity of wave propagation can be easily seen by noting that points of constant phase in the wave advance in space at a rate

$$c = \frac{\omega}{|\mathbf{k}|} \tag{88}$$

A plane-wave solution to the scalar wave equation for elastic media is easily seen to be

$$\Phi(\mathbf{x},\ t) = \Phi_0(\mathbf{x},\ t) = Ae^{\imath(\omega t \pm \mathbf{k}\cdot\mathbf{x})}\ . \tag{89}$$

$$= Ae^{\imath(\omega t \pm (k_x,\ k_y,\ k_z)\cdot(x,\ y,\ z))}\ .$$

To obtain the P-wave displacement field due to the displacement potential Φ, we apply the gradient

$$\mathbf{u} = \bigtriangledown\Phi = \pm iA(k_x\Phi\hat{x} + k_y\Phi\hat{y} + k_z\Phi\hat{z}) \equiv Be^{\imath(\omega t \pm \mathbf{k}\cdot\mathbf{x})}\hat{k}\ . \tag{90}$$

where $B = \pm iA|\mathbf{k}|$. This is a harmonic displacement disturbance with all displacement in the propagation direction $\mp\hat{k}$ (Figure 6a) and an amplitude proportional to $A|\mathbf{k}|$. The change in sign of $\mathbf{u}$ means that the P-wave displacement consists of once-per-wavelength compressions and rarefactions. The dilatation of the P-wave as a function of space and time is

$$\bigtriangledown\cdot\mathbf{u} = \mp A|\mathbf{k}|^2 e^{\imath(\omega t \mp \mathbf{k}\cdot\mathbf{x})}\ , \tag{91}$$

which does not vary perpendicular to the $\hat{k}$ direction. All volumetric strain is this caused by shortening in the $\hat{k}$ direction.

The displacement field from the displacement potential $\mathbf{\Psi}$ is

$$\mathbf{u} = \bigtriangledown\times\mathbf{\Psi} = \bigtriangledown\times(A_x,\ A_y,\ A_z)e^{\imath(\omega t \pm \mathbf{k}\cdot\mathbf{x})} \tag{92}$$

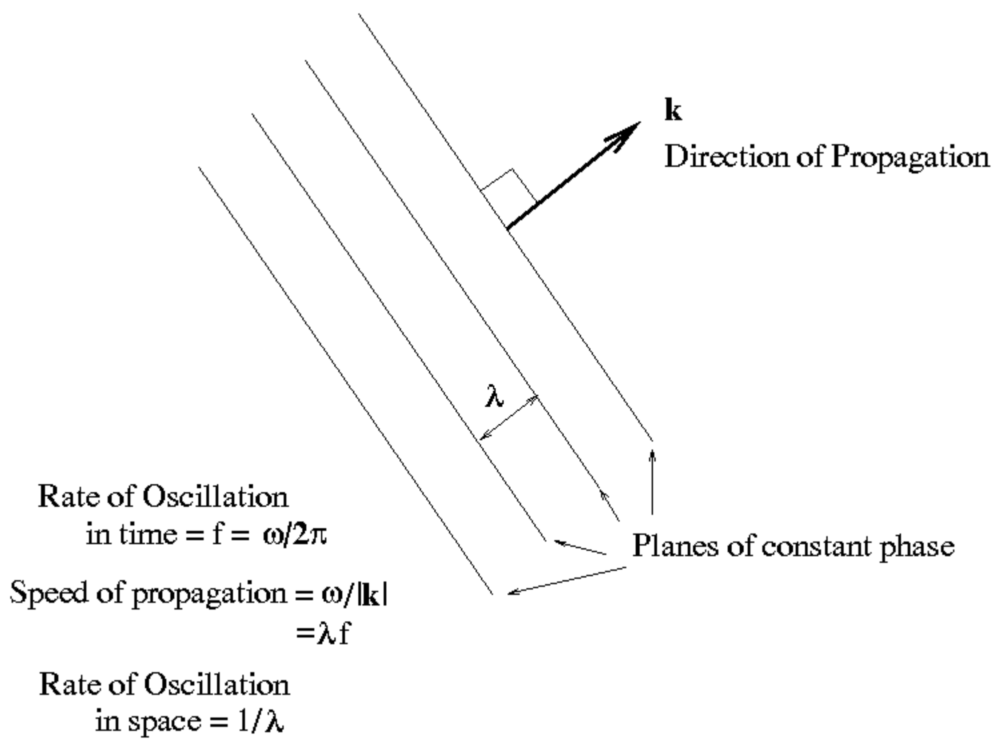

Figure 5: A Plane Wave

$$= \left(\frac{\partial \Psi_z}{\partial y} - \frac{\partial \Psi_y}{\partial z}\right)\hat{x} + \left(\frac{\partial \Psi_x}{\partial z} - \frac{\partial \Psi_z}{\partial x}\right)\hat{y} + \left(\frac{\partial \Psi_y}{\partial x} - \frac{\partial \Psi_x}{\partial y}\right)\hat{z}$$

$$= \pm i\left((A_z k_y - A_y k_z)\hat{x} + (A_x k_z - A_z k_z)\hat{y} + (A_y k_x - A_x k_y)\hat{z}\right) e^{\imath(\omega t \pm \mathbf{k}\cdot\mathbf{x})}$$

The displacement in the S-wave is perpendicular to the propagation direction, $\hat{k}$, and the dilatation, as expected, is zero everywhere and at all times

$$\bigtriangledown \cdot \mathbf{u} = \bigtriangledown \cdot (\bigtriangledown \times \mathbf{\Psi}) = 0 \tag{93}$$

so the shear wave propagates no volumetric strain, only shear strain.

As with transverse elastic waves on a string, seismic waves carry energy in two forms: 1) the strain potential energy in the elastically deformed material; and 2) the kinetic energy of moving material. Consider a shear plane wave propagating in the $\hat{z}$ direction with displacement

$$u_y(z,\ t) = B\cos(\omega t - kz)\ . \tag{94}$$

The kinetic energy per unit volume, V is

$$E_K = \frac{1}{2}\int_V \rho\left(\frac{\partial u_y}{\partial t}\right)^2 dV = \frac{1}{2}\int_V \rho B^2\omega^2 \sin^2(\omega t - kz)\ dV \tag{95}$$

Averaging this density over one wavelength gives the average kinetic energy per unit area of wavefront

$$E_{K\lambda} = \frac{B^2\rho\omega^2}{2\lambda}\int_0^\lambda \sin^2(wt - kz)\ dz = \frac{B^2\rho\omega^2}{4}\ . \tag{96}$$

The potential (strain) energy in a stressed elastic medium can be found by noting that the work done by each increment of strain per unit volume is

$$dW = \sigma_{ij}\ d\epsilon_{ij}\ . \tag{97}$$

If we replace the stress tensor by its constitutive expression (in terms of strain), we have

$$dW = c_{ijkl}\epsilon_{kl}\ d\epsilon_{ij}\ . \tag{98}$$

Integrating over volume, we get the strain energy per unit volume

$$W = \frac{1}{2}c_{ijkl}\epsilon_{kl}\epsilon_{ij} = \frac{1}{2}\sigma_{ij}\epsilon_{ij}\ . \tag{99}$$

For a plain shear wave propagating in the $\hat{z}$ direction with displacement in the $\hat{y}$ direction, the nonzero elements of the strain tensor are

$$\epsilon_{ij} = \epsilon_{ji} = \frac{1}{2}\frac{\partial u_y}{\partial z} = \frac{Bk}{2}\sin(\omega t - kz) \quad (i,j) = (2,3) \tag{100}$$

while the corresponding nonzero stress components for an isotropic medium are

$$\sigma_{ij} = \sigma_{ji} = 2\mu\epsilon_{ij} = B\mu k\ \sin(\omega t - kz) \quad (i,j) = (2,3)\ . \tag{101}$$

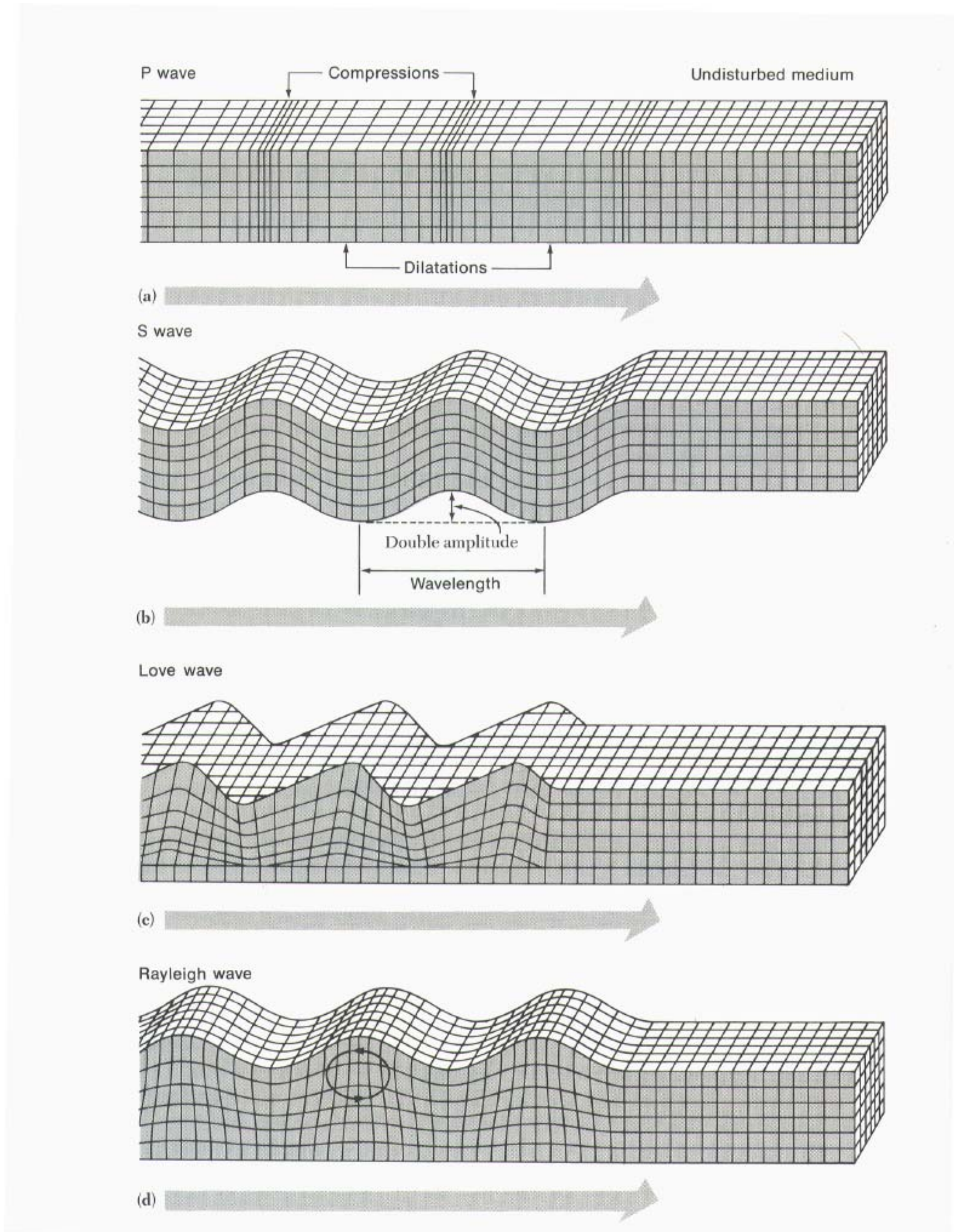

Figure 6: Fundamental Seismic Waves and Their Displacement Fields

The average potential energy per unit area of wavefront is thus

$$E_P = \frac{1}{2\lambda}\int_0^\lambda (\epsilon_{yz}\sigma_{yz} + \epsilon_{zy}\sigma_{zy})\ dz = \frac{B^2\mu k^2}{2\lambda}\int_0^\lambda \sin^2(\omega t - kz)\ dz \tag{102}$$

$$= \frac{B^2\mu k^2}{4} = \frac{B^2\rho\omega^2}{4} = E_K\ .$$

So, as with the elastic string system, we have an equipartition of energy between kinetic and potential types (Figure 3). Also note that, as with the string system, the energy density is proportionate to the density and to the square of amplitude and frequency. The total energy per unit wavefront is thus

$$E_{TS} = E_K + E_P = \frac{B^2\rho\omega^2}{2}\ . \tag{103}$$

Performing the equivalent calculation for P waves propagating in the $\hat{z}$ direction, we now have a plane wave

$$u_z(z,\ t) = A\sin(\omega t - kz)\ . \tag{104}$$

The kinetic energy is essentially the same expression as for the S wave, only the direction of the displacement has changed

$$E_K = \frac{1}{2}\int_V \rho\left(\frac{\partial u_z}{\partial t}\right)^2 dV = \frac{1}{2}\int_V \rho A^2\omega^2\sin^2(\omega t - kz)\ dV \ = \ \frac{A^2\rho\omega^2}{4}\ . \tag{105}$$

To calculate the strain (potential energy) in an equivalent manner, we note that the only non-zero stress tensor element is

$$\sigma_{zz} = (\bar{\lambda} + 2\mu)\epsilon_{zz} = (\bar{\lambda} + 2\mu)\frac{\partial u_z}{\partial z} \tag{106}$$

(where $\bar{\lambda}$ in this context is obviously the Lamé parameter, not the wavelength, λ). The strain energy per unit volume is thus

$$E_P = \frac{1}{2\lambda}\int_0^\lambda \epsilon_{zz}\sigma_{zz}\ dz = \frac{A^2(\bar{\lambda} + 2\mu)k^2}{2\lambda}\int_0^\lambda \cos^2(\omega t - kz)\ dz \tag{107}$$

but

$$\rho\alpha^2 = \bar{\lambda} + 2\mu = \frac{\rho\omega^2}{k^2} \tag{108}$$

so

$$E_P = \frac{A^2\rho\omega^2}{4} \tag{109}$$

which is the exact equation as for S-waves (102) and produces the same result for the total energy

$$E_{TP} = E_K + E_P = \frac{A^2\rho\omega^2}{2}\ . \tag{110}$$

The rate at which energy is carried by seismic waves is given by the energy flux expressions

$$\dot{E}_P = \alpha E_{TP} = \frac{\alpha A^2 \rho \omega^2}{2} \ . \tag{111}$$

and

$$\dot{E}_S = \alpha E_{TS} = \frac{\beta B^2 \rho \omega^2}{2} \ . \tag{112}$$

so the P-wave propagates energy faster by the ratio α/β.

There are, of course, other solutions to the wave equation in homogeneous media. The most important of these is *spherical waves.* We rewrite the scalar wave equation in spherical coordinates (r, θ, ϕ)

$$\bigtriangledown^2 \Phi(\mathbf{r},\ t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} \ . \tag{113}$$

A solution to (113) can be found by trying a solution of the form

$$\Phi(\mathbf{r}, t) = \frac{\xi(\mathbf{r}, t)}{r} \tag{114}$$

which gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left(\frac{r\frac{\partial \xi}{\partial r} - \xi}{r^2}\right)\right) \tag{115}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \xi}{\partial r} - \xi\right) = \frac{1}{rc^2}\frac{\partial^2 \xi}{\partial t^2} \ ,$$

or

$$\frac{1}{r}\left(\frac{r\partial^2 \xi}{\partial r^2} + \frac{\partial \xi}{\partial r} - \frac{\partial \xi}{\partial r}\right) = \frac{\partial^2 \xi}{\partial r^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \tag{116}$$

which is just the 1-dimensional wave equation again, this time in terms of r rather than a Cartesian coordinate such as x! So we have a solution for any function of the form $\xi(\mathbf{r}, t)/r$. Note that this function has a singularity at $r = 0$.