Poles and Zeros

We showed that for any linear system relating two time functions, $x(t)$ and $y(t)$, the frequency-domain response (the transfer function) can be obtained from the governing linear differential equation with constant coefficients of a single variable

$$a_n \frac{d^ny}{dt^n} + a_{n-1} \frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^mx}{dt^m} + b_{m-1} \frac{d^{m-1}x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x. \quad (1)$$

Setting

$$x(t) = e^{i2\pi ft}, \quad (2)$$
$$y(t) = \Phi(f)e^{i2\pi ft}, \quad (3)$$

and solving (1) for $\Phi(f)$ gives the transfer function

$$\Phi(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{j=0}^{m} b_j (2\pi ft)^j}{\sum_{k=0}^{n} a_k (2\pi ft)^k} \equiv \frac{Z(f)}{P(f)} \quad (4)$$

where $Z$ and $P$ are complex polynomials in $f$. The values of $f$ (or equivalently, of the angular frequency $\omega = 2\pi f$) where $Z(f) = 0$ are referred to as zeros of (4), as the response of the system will be zero at those frequencies, no matter what the amplitude of the input. Frequencies for which $P(f) = 0$ are referred to as poles of (4), as the response of the system will be infinite at those frequencies.

In general, the values of $f$ where we have poles and zeros will be complex. If we express the polynomials in (4) in terms of $if$ (or equivalently, $i2\pi f$), then we have real coefficients, and the roots after this change of variables will be real or complex conjugate pairs. It is useful to express the input function (2) as

$$x(t) = e^{i2\pi ft} = e^{i2\pi(f_r + if_i)t} = e^{i2\pi f_r t} \cdot e^{-2\pi f_i t} \quad (5)$$

where $f = f_r + if_i$ and $f_r$ and $f_i$ are real numbers.

This generalized input is:

- A constant for $f = 0$
• A sinusoid for \( f_r \neq 0 \) and \( f_i = 0 \).
• A growing exponential for \( f_r = 0 \) and \( f_i < 0 \)
• A shrinking exponential for \( f_r = 0 \) and \( f_i > 0 \)
• A growing exponentially weighted sinusoid for \( f_r \neq 0 \) and \( f_i < 0 \)
• A shrinking exponentially weighted sinusoid for \( f_r \neq 0 \) and \( f_i > 0 \)

Pole positions are usually displayed graphically in the complex plane using the Laplace transform convention

\[ s = \tau 2\pi f = \sigma + \omega = 2\pi (-f_i + i f_r) \quad . \] (6)

the positions of the poles in \( s \) in the complex plane are an especially useful and compact way to characterize the response of a linear system. Making the substitution (6, and normalizing the leading coefficients of the polynomials, gives us a transfer function expression

\[ \Phi(s) = \frac{Y(s)}{X(s)} = \left(\frac{b_m}{a_n}\right) \frac{\sum_{j=0}^{m} (b_j/b_m)s^j}{\sum_{k=0}^{n} (a_k/a_n)s^k} = K \frac{Z(s)}{P(s)} \] (7)

where \( K = b_m/a_n \) is a scalar gain factor.

The complex roots in \( s \) of the numerator and denominator of (7) will be either real, or will be complex conjugate pairs if the coefficients are real (or equivalently, if the original impulse response is real-valued).

Systems where all pole frequencies have \( \sigma < 0 \) (\( f_i > 0 \), so that the poles lie on the left-hand side of the \( z \) plane, are stable. In this case the only way to get an infinite output is to drive the system with an exponentially increasing sinusoidal input. The impulse response of such a system will always decay back to zero.

On the other hand, systems where all pole frequencies have \( \sigma > 0 \) (\( f_i < 0 \), so that the poles lie on the right-hand side of the \( z \) plane, are unstable; we obtain an infinite output even when the input is exponentially decaying. The impulse response of such systems increases in amplitude with time.

Systems where \( \sigma = 0 \) and \( \omega \neq 0 \) (\( f_i = 0 \) and \( f_r \neq 0 \)) have pole frequency responses that are sinusoidal. Such systems will oscillate forever once they get (even marginally) excited at their resonant frequencies.

Figure (1) shows \( z \)-pole locations and cartoon impulse responses for various 2-pole systems.
Figure 1: Pole locations and system stability for 2-pole systems, real-valued impulse response. Sketched time functions show oscillation and decay characteristics of the corresponding impulse responses.