Introduction to Multidimensional and Multichannel Processing
From a Fourier Perspective

We have now covered many of the widely-used basic tools used in analyzing one-dimensional time or spatial series. Many data sets in geophysics and other physical sciences, however, are inherently multi-dimensional, either because the independent variable is multidimensional (e.g., a 2-dimensional or a 3-dimensional field) or because the time series data itself is a vector quantity (e.g., three-component seismic or electromagnetic data).

Some common examples of such data include photographic records, seismic records from arrays, and gravity and magnetic surveys. Other signals may be considered multidimensional, but with the two axes being physically different, such as a linear array of seismometers, where one dimension is temporal and the other is spatial.

Two-dimensional Spatial Signal Processing.

Let $x(n_1, n_2)$ be a two-dimensional sequence defined for integer $n_1$ and $n_2$. Such a 2-d sequence is usually obtained from sampling a continuous 2-dimensional function. Simple examples of 2-d sequences would be the unit impulse:

$$
\delta(n_1, n_2) = \begin{cases} 
1 & n_1 = n_2 = 0 \\
0 & \text{otherwise}
\end{cases}
$$

(1)

the step function

$$
H(n_1, n_2) = \sum_{m_1 = -\infty}^{n_1} \sum_{m_2 = -\infty}^{n_2} \delta(m_1, m_2) = \begin{cases} 
1 & n_1, n_2 \geq 0 \\
0 & \text{otherwise}
\end{cases}
$$

(2)
the exponential
\[
x(n_1, n_2) = \begin{cases} 
\alpha_1^{n_1} \cdot \alpha_2^{n_2} & n_1, n_2 \geq 0 \\
0 & \text{otherwise}
\end{cases}
\] (3)
and the sinusoid
\[
x(n_1, n_2) = e^{i2\pi(f_1n_1+f_2n_2)}
\] (4)

If a system is linear and time invariant, then convolution is a valid concept in dimensions higher than 1, thus if \(x(n_1, n_2)\) is an input to a two dimensional system which has an impulse response of \(\phi(n_1, n_2)\), then the output is

\[
y(n_1, n_2) = x(n_1, n_2) * \phi(n_1, n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(m_1, m_2)x(n_1-m_1, n_2-m_2)
\] (5)

\[
y = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \phi(n_1 - m_1, n_2 - m_2)x(m_1, m_2)
\] (6)

Direct evaluation of (6) is costly to apply (even more so than direct evaluation of the convolution integral in the 1-d case). Consider a simple case cited by (Rabiner and Gold, 1975), where

\[
\phi(n_1, n_2) = \alpha^{n_1n_2}
\] (7)

and

\[
x(n_1, n_2) = \begin{cases} 
1 & 0 \leq n_1, n_2 \leq 2 \\
0 & \text{otherwise}
\end{cases}
\] (8)

the response, \(\phi(n_1, n_2) * x(n_1, n_2)\) is thus

\[
y(n_1, n_2) = \sum_{m_1=0}^{2} \sum_{m_2=0}^{2} \alpha^{(n_1-m_1)(n_2-m_2)}
\] (9)

which, in general, must be evaluated term by term for each \((n_1, n_2)\) where each term requires \(3^2 = 9\) operations. If \(\phi(n_1, n_2)\) is separable, i.e., it can be written as a product of two 1-d sequences

\[
\phi(n_1, n_2) = g(n_1) \cdot f(n_2)
\] (10)

then the response can be calculated in terms of consecutive 1-dimensional convolutions, and the convolution can be done more rapidly. (6) now becomes

\[
y(n_1, n_2) = \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} g(m_1)f(m_2)x(n_1 - m_1, n_2 - m_2)
\] (11)

\[
y = \sum_{m_1=-\infty}^{\infty} g(m_1) \left( \sum_{m_2=-\infty}^{\infty} f(m_2)x(n_1 - m_1, n_2 - m_2) \right)
\] (12)
where the term inside of the parentheses is a sequence of 1-d convolutions and $m_1$ is allowed to range from $-\infty$ to $\infty$. If the input sequence is also separable, so that $x(n_1, n_2) = a(n_1) \cdot b(n_2)$, then

$$
y(n_1, n_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} g(m_1) f(m_2) a(n_1 - m_1) b(n_2 - m_2) \tag{13}
$$

which is itself separable, i.e.,

$$
y(n_1, n_2) = \alpha(n_1) \cdot \beta(n_2) \tag{15}
$$

where $\alpha(n_1)$ and $\beta(n_2)$ are 1-dimensional convolutions.

As in 1-d systems, sinusoidal inputs play a fundamental role in the understanding of the Fourier response of 2-d systems. This is simply because 2-dimensional sinusoids

$$
x(n_1, n_2) = e^{i2\pi f_1 n_1} e^{i2\pi f_2 n_2} \tag{16}
$$

are eigenfunctions of the 2-d convolution operation, just as 1-d sinusoids were for 1-d systems. Consider the output of a system with impulse response $\phi(n_1, n_2)$ to an input of the form of (16)

$$
y(n_1, n_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \phi(m_1, m_2) e^{i2\pi f_1 (n_1 - m_1)} e^{i2\pi f_2 (n_2 - m_2)} \tag{17}
$$

$$
= e^{i2\pi f_1 n_1} e^{i2\pi f_2 n_2} \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \phi(m_1, m_2) e^{-i2\pi f_1 m_1} e^{-i2\pi f_2 m_2} = x(n_1, n_2) \Phi(f_1, f_2) \tag{18}
$$

where $\Phi(f_1, f_2)$ is the frequency response of the system in two dimensions and hence defines a 2-d Fourier transform of a 2-d sampled function. The corresponding inverse transformation is just

$$
\phi(n_1, n_2) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \Phi(f_1, f_2) e^{i2\pi f_1 n_1} e^{i2\pi f_2 n_2} df_1 df_2. \tag{19}
$$

Note that $\Phi(f_1, f_2)$ is periodic in frequency with unit period along both the $f_1$ and $f_2$ axes, as we’d expect for a sampled function with a uniform sampling interval, so that

$$
\Phi(f_1, f_2) = \phi(f_1 + l, f_2 + m) \quad (l, m) \text{ integers} \tag{20}
$$

which is two-dimensional aliasing. If $\phi(n_1, n_2)$ is real, then

$$
\Phi(f_1, f_2) = \sum_{m_1 = -\infty}^{\infty} \sum_{m_2 = -\infty}^{\infty} \phi(m_1, m_2) e^{-i2\pi f_1 m_1} e^{-i2\pi f_2 m_2} = \Phi^*(-f_1, -f_2) \tag{21}
$$
so that $\Phi(f_1, f_2)$ is Hermitian in a 2-d sense.

We now have the tools for performing windowed filter design in 2-dimensions. Consider the perfect low pass filter with a response given by

$$
\Phi(f_1, f_2) = \left\{ \begin{array}{ll}
1 & -\alpha \leq f_1 \leq \alpha, -\beta \leq f_2 \leq \beta \\
0 & \text{otherwise}
\end{array} \right. \quad (22)
$$

taking the inverse Fourier transform gives the $n$ (usually spatial) domain series

$$
\phi(n_1, n_2) = \int^{-\alpha}_{-\alpha} \int^{-\beta}_{-\beta} e^{i2\pi f_1 n_1} e^{i2\pi f_2 n_2} df_2 df_1 \quad (23)
$$

If the frequency response is separable, so is the $n$ domain response, so

$$
\phi(n_1, n_2) = \left( \int^{-\alpha}_{-\alpha} e^{i2\pi f_1 n_1} df_1 \right) \left( \int^{-\beta}_{-\beta} e^{i2\pi f_2 n_2} df_2 \right) \quad (24)
$$

$$
= \left( \frac{e^{i2\pi f_1 n_1}}{i2\pi n_1} \right|_{-\alpha}^{\alpha} \left( \frac{e^{i2\pi f_2 n_2}}{i2\pi n_2} \right|_{-\beta}^{\beta} = \left( \frac{\sin(2\pi \alpha n_1)}{\pi n_1} \right) \left( \frac{\sin(2\pi \beta n_2)}{\pi n_2} \right) . \quad (25)
$$

This frequency response and a plot of its corresponding FIR filter weights is shown in Figure 1

Unless we have a physical reason for treating the $n_1$ and $n_2$ directions unequally, to low-pass filter a 2-d sequence, we generally want to have a response which is circularly symmetric in the time and frequency domains. Such a filter is specified by

$$
\Phi(f_1, f_2) = \left\{ \begin{array}{ll}
1 & f_1^2 + f_2^2 \leq f_{max}^2 \\
0 & \text{otherwise}
\end{array} \right. \quad (26)
$$
and the corresponding filter weights are obtainable from (19) as
\[
\phi(n_1, n_2) = \int_{-f_{\text{max}}}^{f_{\text{max}}} \int_{-(f_{\text{max}} - f_{n_2}^2)^{1/2}}^{(f_{\text{max}} - f_{n_1}^2)^{1/2}} e^{i2\pi f_1 n_1} e^{i2\pi f_2 n_2} df_2 df_1 \tag{27}
\]
An easy way to evaluate this integral is to note that both the \(n\) and frequency response of the system are circularly symmetric, thus, we can obtain the general solution by finding \(\phi(n_1, 0)\) and then substituting \((n_2^2 + n_2^2)^{1/2}\) for \(n_1\).
\[
\phi(n_1, 0) = \int_{-f_{\text{max}}}^{f_{\text{max}}} \int_{-(f_{\text{max}} - f_{n_1}^2)^{1/2}}^{(f_{\text{max}} - f_{n_1}^2)^{1/2}} e^{i2\pi f_1 n_1} df_2 df_1 \tag{28}
\]
\[
= \int_{-f_{\text{max}}}^{f_{\text{max}}} e^{i2\pi f_1 n_1} \cdot 2(f_{\text{max}}^2 - f_{n_1}^2)^{1/2} df_1 \tag{29}
\]
using the polar substitution \(f_1 = f_{\text{max}} \sin \theta\) gives
\[
= \int_{-\pi/2}^{\pi/2} 2(f_{\text{max}}^2 - f_{\text{max}}^2 \sin^2 \theta)^{1/2} e^{i2\pi f_{\text{max}} n_1 \sin \theta} \cdot f_{\text{max}} \cos \theta d\theta \tag{30}
\]
\[
= 2f_{\text{max}}^2 \int_{-\pi/2}^{\pi/2} \cos^2 \theta e^{i2\pi f_{\text{max}} n_1 \sin \theta} d\theta \tag{31}
\]
\[
= 2\pi f_{\text{max}} J_1(2\pi f_{\text{max}} n_1) \tag{32}
\]
where \(J_1\) is the first-order Bessel function. Thus
\[
\phi(n_1, n_2) = \frac{2\pi f_{\text{max}} J_1(2\pi f_{\text{max}} (n_1^2 + n_2^2)^{1/2})}{(n_1^2 + n_2^2)^{1/2}} \tag{33}
\]
which is plotted in Figure 2.

Before proceeding further with the topic of 2-d filtering, we shall define a 2-dimensional DFT. The utility of the multidimensional DFT arises for the same reasons as for 1-d series; it enables us to deal with limited extent sampled time series (with the added implication that our sampled signals are now periodic), and it is implementable with highly efficient FFT routines.

A periodic signal in two dimensions satisfies
\[
x(n_1, n_2) = x(n_1 + m_1 N_1, n_2 + m_2 N_2) \tag{34}
\]
where \((N_1, N_2)\) are the periods of the 2-d signal (in samples) along the two grid axes and
\[
(m_1, m_2) \text{ integers} \tag{35}
\]
As in 1-dimensional analysis, 2-d signals can be decomposed into a linear combination of a finite number of exponential basis functions which have periods which are submultiples of \((N_1, N_2)\). Thus,
\[
x(n_1, n_2) = \frac{1}{N_1 N_2} \sum_{k_1 = 0}^{N_1 - 1} \sum_{k_2 = 0}^{N_2 - 1} X(k_1, k_2) e^{i2\pi n_1 k_1 / N_1} e^{i2\pi n_2 k_2 / N_2} \tag{36}
\]
Circular Low Pass 2-d Spatial Filter

Frequency Response  Spatial Response

Figure 2: Response and corresponding FIR weights for a circular frequency response low-pass filter; \( f_{\text{max}} = f_s/4 \); 32 by 32 points.

where \( X(k_1, k_2) \) is the 2-d DFT of \( x(n_1, n_2) \). The corresponding IDFT is therefore

\[
X(k_1, k_2) = \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x(n_1, n_2) e^{-i2\pi n_1 k_1/N_1} e^{-i2\pi n_2 k_2/N_2}.
\]

(37)

We can also define a 2-d Z transform

\[
x(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) z_1^{-n_1} z_2^{-n_2}
\]

(38)

and a corresponding inverse Z transform

\[
x(n_1, n_2) = \frac{1}{(i2\pi)^2} \int_{c_1} \int_{c_2} X(z_1, z_2) z_1^{n_1-1} z_2^{n_2-1} \, dz_1 \, dz_2.
\]

(39)

A general 2-d digital filter is thus characterizable by a difference equation

\[
y(n_1, n_2) = \sum_{i=-p}^{p} \sum_{j=-q}^{q} \alpha_{ij} x(n_1 - i, n_2 - j) - \sum_{i=-r}^{r} \sum_{j=-s}^{s} \beta_{ij} y(n_1 - i, n_2 - j)
\]

(40)

where \( i \) and \( j \) are not both zero in the second summation, and we have made the constant coefficients symmetric about \( g(0, 0) \). (40) has a Z transform given by

\[
Y(z_1, z_2) = \sum_{i=-p}^{p} \sum_{j=-q}^{q} \beta_{ij} z_1^{-i} z_2^{-j}
\]

\[
\sum_{i=-r}^{r} \sum_{j=-s}^{s} \alpha_{ij} z_1^{-i} z_2^{-j}
\]

(41)

where \( \alpha_{00} = 1 \). (41) has poles and zeros in a 4-dimensional space defined by the real and imaginary parts of \( z_1 \) and \( z_2 \). Evaluating stability for such filters
is difficult, primarily because one cannot, in general, factor the 2-dimensional numerator and denominator of (41) to obtain a simple view of the zero and pole frequencies. As a result of this property (or non-property) of higher-dimensional polynomials, the cascade of two stable 2-dimensional IIR filters may not even be stable! Because of these difficulties, we will primarily concern ourselves with FIR filters in this introduction (which have no poles and thus no potential stability problems).

Consider a circularly frequency-symmetric, low-pass filter case defined by the ideal response

\[ \Phi(f_1, f_2) = \begin{cases} 1 & f_1^2 + f_2^2 \leq 1/4 \\ 0 & \text{otherwise} \end{cases} \]  

(42)

from (33), we know that the corresponding filter weights are given by

\[ w(n_1, n_2) = \frac{\pi f_{\text{max}} J_1((\pi/2)(n_1^2 + n_2^2)^{1/2})}{2(n_1^2 + n_2^2)^{1/2}} \]  

(43)

Taking an \( N \) by \( N \) -point rectangular window (a simple truncation of the 2-d series) produces a filter with a frequency response

\[ W(f_1, f_2) = \sum_{n_1=-[(N-1)/2]}^{[(N-1)/2]} \sum_{n_2=-[(N-1)/2]}^{[(N-1)/2]} w(n_1, n_2)e^{-i2\pi n_1 f_1}e^{-i2\pi n_2 f_2} \]  

(44)

where \((f_1, f_2)\) is normalized to the Nyquist interval, so that both frequencies span \((-1/2, 1/2)\).

As in the 1-d case, we can improve this response considerably by applying a windowing function with better spectral leakage characteristics than the rectangular window. As we usually wish our window to be circularly symmetric in the \((n_1, n_2)\) planes, we can take a window function, \( \hat{w} \), from 1-dimensional analysis and substitute the radius in \((n_1, n_2)\)-space for \( n \) to obtain a 2-dimensional window

\[ w(n_1, n_2) = \hat{w}((n_1^2 + n_2^2)^{1/2}) \]  

(45)

As in 1-d processing, the Kaiser-Bessel window is a good candidate for a windowing function due to its low spectral leakage. Using (45), an \( N \) by \( N \), 2-d Kaiser-Bessel window may be defined by

\[ w(n_1, n_2) = \frac{I_0 \left[ 2\pi \sqrt{1 - (n_1^2 + n_2^2)/N^2} \right]}{I_0(2\pi)} \]  

(46)

for \( n_1^2 + n_2^2 \leq N^2 \) and

\[ w(n_1, n_2) = 0 \]  

(47)

for \( n_1^2 + n_2^2 > N^2 \), where \( I_0(x) \) is the modified Bessel function of the first kind and 0th order. The response of the Kaiser-Bessel windowed low pass filter (Figure 3) is superior in smoothness and in attenuation to the rectangular window (Figure 4).
2-d Kaiser–Bessel Window

Spatial Domain         Frequency domain

Figure 3: A 2-d Kaiser-Bessel window and its spectrum.

Circular Low Pass 2-d Spatial Filter

Rectangular Window      2-d K-B Window

Figure 4: FIR filter response for a circular frequency response low-pass filter; $f_{\text{max}} = f_s/4$, with FIR weights windowed by a rectangular window, and by a Kaiser-Bessel window (Figure 3). Note the usual tradeoff between ripple and rapidity of cut-off.
Frequency-Wavenumber Filtering

We next consider multi-dimensional systems where the dimensions do not have the same units, but rather have observables that depend on both space and time. Consider a linear array of seismometers or antennae deployed in the $\hat{x}$ direction with a constant spacing. Signals from such an array can be displayed in a 2-dimensional record section, where we have $t$ as the ordinate and channel number, or $x$, as the abscissa (or vice-versa). The response of such a system to a traveling, sinusoidal plane wave of frequency $f_0$

$$\phi(t, x) = e^{i2\pi f_0(t - x/v_0)} = e^{i2\pi f_0(t - x/v_0)} \quad (48)$$

where $v_0$ is the apparent velocity of the wave across the array. Apparent velocities are of particular interest, as such signals impinge upon the array at specific angles given by

$$\theta = \sin^{-1}(c/v_0) \quad (49)$$

where $c$ is the wave velocity in the medium and $\theta$ is the angle between the planar wavefront and the $\hat{x}$ direction. Thus, when $\theta = 0$, the apparent velocity $v_0 = \infty$, as the wavefront strikes all of the sensors simultaneously. Conversely, when $\theta = 90^\circ$, $v_0 = c$, as the plane wave is propagating directly along the array axis (in the $\hat{x}$ direction).

If we arrange the data in ($t, x$)-space to form a regular 2-dimensional array (practically speaking, we may have to resample the traces to form an evenly-spaced array in the sampled case), we can take a 2-d Fourier transform of (48) as

$$\Phi(f, k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t, x) e^{-i2\pi ft} e^{i2\pi kx} dt \, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t, x) e^{-i2\pi ft} e^{i2\pi kx} dt \, dx \quad (50)$$

where the wavenumber (or spatial frequency) is a reciprocal length defined as

$$k = \frac{1}{\lambda} = \frac{f}{v} = \frac{\omega}{2\pi v} \quad . \quad (51)$$

The $f - k$ transform of the plane wave evaluate (48) is thus

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i2\pi f_0 t} e^{-i2\pi k_0 x} e^{-i2\pi f t} e^{i2\pi kx} dt \, dx = \delta(f - f_0, k - k_0) \quad (52)$$

so that sinusoidal traveling wave components with given frequencies and wavenumbers in ($x, t$)-space map to delta functions in ($f, k$)-space.

Note that we have chosen a mixed sign convention for the forward $f - k$ transform, where the frequency portion has a minus sign in the exponent, consistent with our previous convention for 1- and 2-dimensional transforms, but the wavenumber transform exponent has a plus sign. We do this so that traveling wave components which propagate towards increasing $x$ for increasing $t$ (like 48) map into the first quadrant of the $f - k$ plane.
Figure 5: Basic mappings for f-k filtering from a linear array.
In $f-k$ space, arbitrary signals of a given apparent velocity, $v_0$ are specified by (51), so that such signals in $f-k$ space lie along lines which intersect the $f-k$ origin and have slopes of $v_0$ in $\omega-k$ space (Figure 5).

Now suppose that we wish to selectively resolve waves within a range of apparent velocities. This procedure is called beam forming, as it was first developed in radar and radio transmission applications. Because of Snell’s law, the horizontal phase velocity of a signal remains constant throughout a given ray path in a horizontally homogeneous medium. Thus, beam forming using seismic array data selectively filters waves which turn within a particular depth range (as a seismic array is generally deployed horizontally).

For a 1-d array of sensors we could, for example, remove signal components with low apparent velocities and conversely preferentially extract signals with higher apparent velocities (say above some corner value, $v_0$) by employing a filter with an $f-k$ response given by

$$Y(f,k) = \begin{cases} 1 & -\frac{|f|}{v_0} \leq k \leq \frac{|f|}{v_0} \\ 0 & \text{otherwise} \end{cases}$$

(53)

It’s instructive to examine the impulse response of (53) (e.g., Kanasewich, 1975), given by the inverse $f-k$ transform

$$y(t, x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Y(f,k) e^{i2\pi ft} e^{-i2\pi kx} dk df.$$  

(54)

Of course, in practical situations, $x$ and $t$ are both discrete variables, so that (54) becomes (for unit time sampling interval, $\Delta t = 1$ and unit spatial sampling interval, $\Delta x = 1$)

$$y(n\Delta t, (m+1/2)\Delta x) = y(n, m+1/2) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} Y(f,k) e^{i2\pi fn} e^{-i2\pi k(m+1/2)} dk df$$

(55)

where we have assumed that there are an even number of receivers in the array, so that the half-integer spatial index, $m + 1/2$ gives a symmetric deployment relative to the $x$ origin. Evaluating the integral over $k$ for $Y(f,k)$ defined by (53) gives

$$\int_{-1/2}^{1/2} e^{i2\pi fn} \left( \frac{e^{-i2\pi k(m+1/2)}}{-i(m+1/2)} \right) \frac{|f|/v_0}{|k|/v_0} df$$

(56)

$$= \frac{1}{\pi(m+1/2)} \int_{-1/2}^{1/2} e^{i2\pi fn} \sin(2\pi(m+1/2)|f|/v_0) df$$

(57)

$$= \frac{2}{\pi(m+1/2)} \int_{0}^{1/2} \cos 2\pi fn \sin(2\pi(m+1/2)f/v_0) df$$

(58)

using unity apparent velocity as the cutoff value for the sake of illustration gives

$$v_0 = \frac{\Delta x}{\Delta t} = 1$$

(59)
so that

\[ y(n, m + 1/2) = \frac{2}{\pi(m + 1/2)} \int_0^{1/2} \sin(2\pi f(m + 1/2)) \cos(2\pi fn) df. \tag{60} \]

As

\[ \int \sin(mx) \cos(nx) \, dx = -\frac{\cos(mx - nx) \times (m^2 - n^2)}{2(m - n) \times 2(m + n)} \quad (m^2 \neq n^2) \tag{61} \]

(60) gives

\[ y(n, m + 1/2) = \frac{2}{\pi(m + 1/2)} \left( -\frac{\cos(2\pi f(n + m + 1/2))}{4\pi(n + m + 1/2)} - \frac{\cos(2\pi f(-n + m + 1/2))}{4\pi(-n + m + 1/2)} \right) \bigg|_0^{1/2} \tag{62} \]

\[ = \frac{2}{\pi(m + 1/2)} \left( -\frac{\cos(\pi(n + m + 1/2))}{4\pi(n + m + 1/2)} - \frac{\cos(\pi(-n + m + 1/2))}{4\pi(-n + m + 1/2)} + \frac{1}{4\pi(n + m + 1/2)} + \frac{1}{4\pi(-n + m + 1/2)} \right). \tag{63} \]

As \( m \) and \( n \) are integers, the cosine terms are zero, so that

\[ y(n, m + 1/2) = \frac{1}{2\pi^2(m + 1/2)} \left( \frac{1}{(n + m + 1/2)} + \frac{1}{(-n + m + 1/2)} \right) \tag{65} \]

or

\[ y(n, m + 1/2) = \frac{1}{\pi^2 [(m + 1/2)^2 - n^2]} \tag{66} \]

As is usual in FIR filter design problems, the weights (66) are nonzero for all \( n \) and \( m \) and we are thus forced to consider truncation issues. As previously noted, the Kaiser-Bessel window provides a good choice for truncating 2-d weights. Rectangular and Kaiser-Bessel windowed realizations of the velocity filter (66) for 64 channels of 64 sample data are shown in Figures 6 and 7.

As the 3-d perspective plots make it difficult to see the \( x-t \) domain impulse response, each time series in the impulse response consists of a simple convolving kernel. The response of the filter, \( r(n, m + 1/2) \) to an arbitrary input, \( \phi(n, m + 1/2) \), is thus given by the 2-d convolution of (66) with the input traces

\[ r(n, m + 1/2) = \sum_{i=1}^{N} \sum_{j=-M/2}^{M/2-1} \phi(i, j + 1/2) y(n - i, m + 1/2 - j) \tag{67} \]

\( r(n, 1/2) \) is thus obtainable by convolving each time series in the input with the corresponding time series in the impulse response, followed by a summation (or stack) of the resultant \( M \) convolutions all \( m \).

A particularly simple \( f - k \) filter has weights given by

\[ y(n, m) = \delta(0, m) \tag{68} \]
Figure 6: Impulse and frequency response for a high-apparent velocity filter; rectangular windowing.

Figure 7: Impulse and frequency response for a high-apparent velocity filter; Kaiser-Bessel windowing.
Zero-Lag stack \((N,M = 64)\)

\[
Y(\nu, \mu) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \delta(0, m)e^{-i2\pi \nu n/N} e^{i2\pi \mu m/M}
\]

where the frequency-wavenumber indices are the integers \((\nu, \mu)\). Thus,

\[
Y(\nu, \mu) = \sum_{m=0}^{M-1} e^{i2\pi \mu m/M} = \frac{1 - e^{i2\pi \mu}}{1 - e^{i2\pi \mu/M}}
\]

which has the amplitude response of the Dirichlet kernel

\[
|Y(\nu, \mu)| = \frac{\sin(\pi \mu)}{\sin(\pi \mu/M)}
\]

and is independent of the Nyquist-normalized frequency, \(\nu\). The \(t-x\) and \(f-k\) plots for the zero-lag stack are shown in Figure 8.

The zero-lag stack, then, acts like a low pass filter in \(k\) and a high pass filter in \(v\) (or, equivalently, and all-pass filter in \(f\)), so that waves with large \(k\) (short wavelengths) and low \(v\) (less vertical ray paths) will be attenuated, while those with small \(k\) (long wavelengths) will be relatively unaffected, all other things being equal). The filter operates best on passing plane waves that arrive at all receivers simultaneously (with infinite apparent velocity).

Consider next what happens if we stack the time series with some time lag, \(\Delta\), imposed between the channels, so that the impulse response is now

\[
y(n, m) = \delta(n + \Delta m)
\]
Zero-Lag stack \((N,M = 64)\)

\[ Y(\nu, \mu) = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \delta(n + \Delta m) e^{-i2\pi \nu n/N} e^{i2\pi \mu m/n} \] (73)

\[ = \sum_{m=0}^{M-1} e^{i2\pi \nu \Delta m/N} e^{i2\pi \mu m/M} \] (74)

for the symmetric case \(N = M\), we have

\[ = \sum_{m=0}^{M-1} e^{i2\pi m(\nu \Delta + \mu)/N} \] (75)

which gives the amplitude response

\[ |Y(\nu, \mu)| = \frac{\sin(\pi(\nu \Delta + \mu))}{\sin(\pi(\nu \Delta + \mu)/M)} \] (76)

shown in Figure 9 for \(\Delta = 1\), along with the response of the unlagged series (71).

Rotating the impulse response in the \(t-x\) domain by applying a constant lag between equally-spaced sensors has thus simply rotated the Fourier Transform by the same angle (in this case, 45 degree). This observation could be applied to modify the velocity filter to enclose some other hourglass-shaped swath of the \(f-k\) plane – one would simply impose a linear lag between the initial time traces of the form of (69) to rotate the response function to the desired angle.
An important application of phased arrays is to receive or transmit narrow frequency band energy preferentially from a small range of azimuths. Consider a linear hydrophone array trailed from a ship with an array element spacing of $\Delta x = 30$ m and a length of 3600 m ($M = 121$ elements in all). If such an array is receiving energy from a narrow-band source (so that we are only interested in a small range of frequencies), we can calculate the width of the main lobe of the Dirichlet kernel response if we know the sound speed (about 1500 m/s in water).

For a $f_1 = 50$ Hz source, the wavelength is thus about 30 m. The $f-k$ response (Figure 10) of such a streamer for stacked traces is (from 71)

$$|Y(\nu, \mu)| = \frac{\sin(\pi \mu)}{\sin(\pi \mu / M)}$$ (77)

where we can convert a general discrete $f-k$ transform to a function of Nyquist-normalized wavenumber, $k$, and Nyquist-normalized frequency, $f$, using the transformations

$$\mu = M k / k_s$$ (78)
$$\nu = N f / f_s$$ (79)

where $k_s$ is the spatial sampling frequency

$$k_s = 1 / \Delta x = 1 / 30 \text{ m}^{-1}$$ (80)

and $f_s$ is the time sampling frequency to obtain

$$|Y(f, k)| = \frac{\sin(M \pi k / k_s)}{\sin(\pi k / k_s)}.$$ (81)

This expression has its first zero at $k = k_1$, defined by

$$\sin(M \pi k_1 / k_s) = 0 \quad (k_1 \neq 0)$$ (82)

or where

$$k_1 = k_s / M \approx 2.75 \times 10^{-4} \text{ m}^{-1}$$ (83)

which occurs at a plane wave emergence angle of

$$\theta = \sin^{-1}(c / v_1) = \sin^{-1}(ck_1 / f_1) = \sin^{-1}(1500 \cdot 2.75 \times 10^{-4} / 50) \approx 0.47^\circ$$ (84)

(corresponding to a phase lag of 2$\pi$ between the first and last hydrophones) so that the total width of the main lobe is $\pm \theta$, or about 1$^\circ$. The second major maximum occurs when the contributions of the plane wave are again in phase at all of the receivers, where $k = k_s$ and

$$\theta = \sin^{-1}(ck_s / f_1) = \sin^{-1}(1500 / (30 \cdot 50)) = 90^\circ.$$ (85)

in this case. If the frequency were doubled to $f_2 = 100$ Hz, then the wavelength would be halved, and the main lobe would become narrower, with the first zero now occurring at

$$\theta = \sin^{-1}(ck_1 / f_2) = \sin^{-1}(1500 \cdot 2.75 \times 10^{-4} / 100) \approx 0.24^\circ.$$ (86)
Figure 10: Array response for 121 receivers at a spacing of 30 m to a 50 Hz wave with a phase velocity of 1500 m/s.
In this case, the second major maximum now occurs at only
\[ \theta = \sin^{-1}(ck_x/f_2) = \sin^{-1}(1500 \cdot 2/(60 \cdot 100)) = 30^\circ. \] (87)
so that the main beam has become narrower, but we now have a second maximum to contend with at 30 degrees from normal incidence.

**Frequency-Wavenumber Filtering with 2-dimensional arrays.**

Next, we consider data from a 2-dimensional array of instruments. Again, we can decompose incident energy into a superposition of traveling waves, but we now have an additional spatial dimension to contend with.

A particular wave field sampled by a two-dimensional array can be decomposed into plane waves
\[ \phi(t, x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(f, k_x, k_y) e^{i2\pi ft} e^{-ik_x x} e^{-ik_y y} df \, dk_x \, dk_y \] (88)
where \( k_x \) and \( k_y \) are the wavenumbers in the \( x \) and \( y \) directions and \( \Phi(k_x, k_y, f) \) is a 3-dimensional frequency-wavenumber spectrum. A particular plane wave propagates at an azimuthal angle, \( \phi \), specified by
\[ \phi = \tan^{-1}(k_y/k_x) \] (89)
\( k_x \) and \( k_y \) are thus not independent, but are related by the Pythagorean theorem
\[ k_x^2 + k_y^2 = f^2/v^2. \] (90)
The \( f-k \) spectrum of a 2-dimensional time signal is thus
\[ \Phi(f, k_x, k_y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(t, x,y) e^{-i2\pi ft} e^{ik_x x} e^{ik_y y} dt \, dx \, dy \] (91)
and its discrete counterpart is
\[ \Phi(\nu, \mu_x, \mu_y) = \sum_{n=1}^{N-1} \sum_{l=1}^{L-1} \sum_{m=1}^{M-1} \phi(l, n,m) e^{-i2\pi n\nu/N} e^{i2\pi l\mu_x/L} e^{i2\pi m\mu_y/M}. \] (92)
As in the case of an ideal 1-dimensional array, we can (theoretically at least) calculate a frequency-wavenumber spectrum from real data using (92) to determine the nature of the incident energy in terms of a plane wave decomposition. Unfortunately, this is not usually the case in seismology, particularly at high frequencies, as spatial heterogeneity induces scattering which fragments the wavefront near the array, reducing the signal coherence from sensor to sensor. One can improve the situation somewhat by introducing station corrections (e.g., Aki and Richards, p. 610), so that the wavefront is best reconstructed (this procedure is analogous to the adaptive optical techniques used in modern large
telescopes). The arrival time of a particular phase at the instrument located at 
$(x_i, y_i)$ is then

$$t_i = t_0 + \frac{\cos \phi}{v}(x_i - x_0) + \frac{\sin \phi}{v}(y_i - y_0) + \tau_i$$

(93)

where $t_0$, $x_0$, and $y_0$ are all reference coordinates and $\tau_i$ is the station correction. Of course, we may not have a priori knowledge of the apparent velocity, $v$ the ray azimuth, $\phi$, and/or the station correction, $\tau_i$. What has been done in practice is to form all possible beams over a range of station corrections and select the one which gives the best signal-to-noise ratio.

In general, the ability of any array to beamform at a given frequency may be characterized by (92), or by its corresponding power spectral density estimate, $|\Phi(\nu, \mu_x, \mu_y)|^2$. 