Plane Waves in Layered Media

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We now address the next level of complexity in understanding seismic wave propagation, where we consider the Earth (or at least an insightful subregion of it) to consist of isotropic, homogeneous flat layers, stacked in the $\hat{z}$ direction, with welded interfaces, where by "welded" we mean that all tractions and displacements are continuous (Figure 1).

We will first consider plane waves with displacements which are purely in the $\hat{x}$ and/or $\hat{z}$ directions. These will be P waves with the displacement potential

$$\Phi(x,\ z,\ t) = Ae^{\imath(\omega t - k_{x,\alpha}x \pm k_{z,\alpha}z)} \tag{1}$$

and S waves with the displacement potential

$$\mathbf{\Psi}(x,\ z,\ t) = \mathbf{B}e^{\imath(\omega t - k_{x,\beta}x \pm k_{z,\beta}z)} \tag{2}$$

where the vector wavenumbers are

$$\mathbf{k}_\alpha = (k_{x,\alpha},\ k_{z,\alpha}) \tag{3}$$

$$\mathbf{k}_\beta = (k_{x,\beta},\ k_{z,\beta}) \tag{4}$$

$$|\mathbf{k}_\alpha| = \sqrt{k_{x,\alpha}^2 + k_{z,\alpha}^2} = \frac{\omega}{\alpha} \tag{5}$$

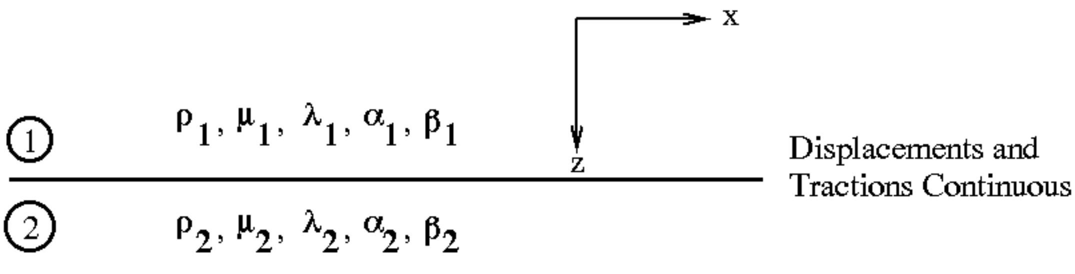

Figure 1: A Two-Layer Elastic Medium

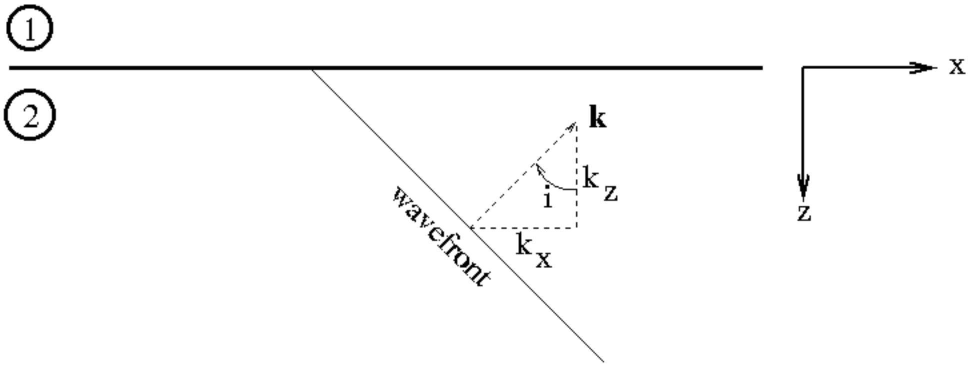

Figure 2: A Wavefront Impinging upon a Welded Interface

and

$$|\mathbf{k}_\beta| = \sqrt{k_{x,\beta}^2 + k_{z,\beta}^2} = \frac{\omega}{\beta} \ . \tag{6}$$

Note that k_x is the same for all of these expressions (we shall see later that this is required by *Snell's law*). Traction and displacement continuity conditions across interfaces will result in coupling between these P and S waves, so that impinging P waves can generate S waves at the interface and vice-versa. The system of P and S plane waves with propagation directions and displacements in the (x, z) plane (the plane of the page for Figure 2) is called the *$P-S_V$ system.* Consider a wavefront characterized by k_x and k_z arriving at the interface with an incidence angle of

$$i = \tan^{-1}(k_x/k_z) = \sin^{-1} k_x/|\mathbf{k}| = \cos^{-1} k_z/|\mathbf{k}| \tag{7}$$

(not to be confused with $\imath = \sqrt{-1}$, of course). The *apparent velocity* of the wavefront is its velocity parallel to the interface

$$c_x = \frac{c}{\sin i} = \frac{\omega}{|\mathbf{k}| \sin i} = \frac{\omega}{k_x} \ . \tag{8}$$

Defining

$$\frac{k_{z,\alpha}}{k_{x,\alpha}} = \cot i = r_\alpha \tag{9}$$

and

$$\frac{k_{z,\beta}}{k_{x,\beta}} = \cot j = r_\beta \ , \tag{10}$$

where i is the angle of incidence for a P wave and j is the angle of incidence for an S wave, we can now write the system potentials in terms of the horizontal wavenumber, k_x, alone

$$\Phi(x,\ z,\ t) = A e^{\imath(\omega t - k_{x,\alpha} x \pm k_{x,\alpha} r_\alpha z)} \tag{11}$$

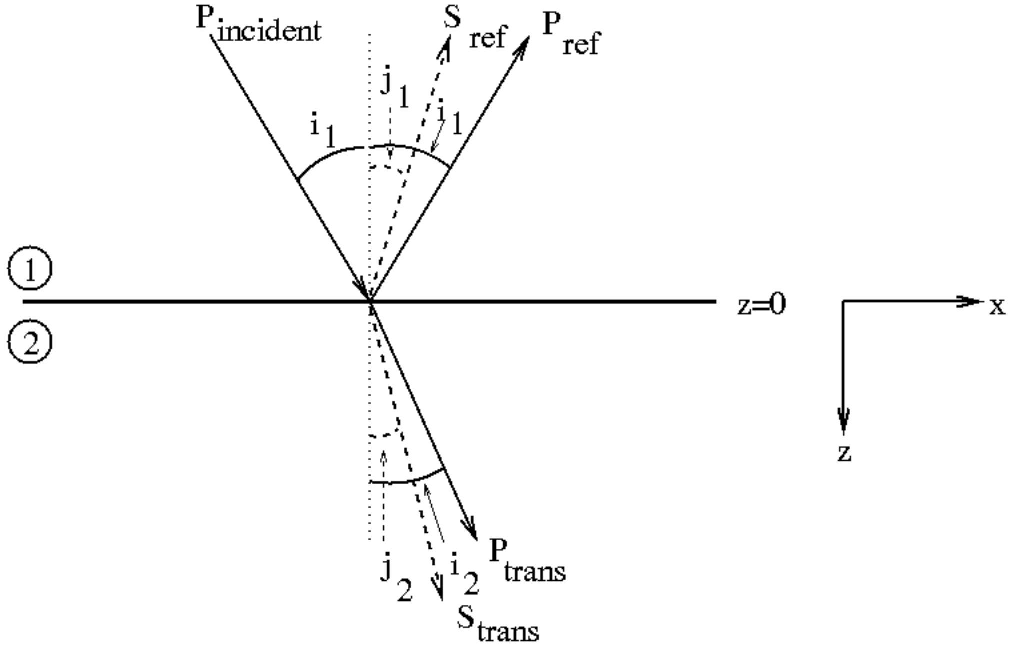

Figure 3: An Incident P wave and Converted Waves, P-$\mathrm{S_V}$ System

$$\mathbf{\Psi}(x,\ z,\ t) = \mathbf{B}e^{\imath(\omega t - k_{x,\beta}x \pm k_{x,\beta}r_\beta z)}\ . \tag{12}$$

To investigate how these various plane waves interact, we shall derive *Snell's Law* which controls their propagation directions. Consider the plane wave systems for P and S waves incident from medium 1 (Figures 3 and 4).

Because displacements and tractions are continuous across a welded interface (as, indeed, they are everywhere else within a connected medium), so must the horizontally traveling argument, $\omega t \pm k_x x$ of a traveling plane wave be continuous. Just as with the transverse waves on a string, we have the excitation in medium 2 being created by the excitation in medium 1. Wavefronts must thus stay in step along the boundary (Figure 5) by having the same apparent velocity, c_x (8), regardless of what wavetype (P,S) or which medium (1,2) we are considering.

Furthermore, the frequency of the harmonic wave must stay the same on either side to keep the wavefronts in lock step along the interface. Because $k_x = \omega / c_x$, we thus have

$$k_{x1,\alpha} = k_{x1,\beta} = k_{x2,\alpha} = k_{x2,\beta} \tag{13}$$

$$= \frac{\alpha_1}{\sin i_1} = \frac{\beta_1}{\sin j_1} = \frac{\alpha_2}{\sin i_2} = \frac{\beta_2}{\sin j_2}$$

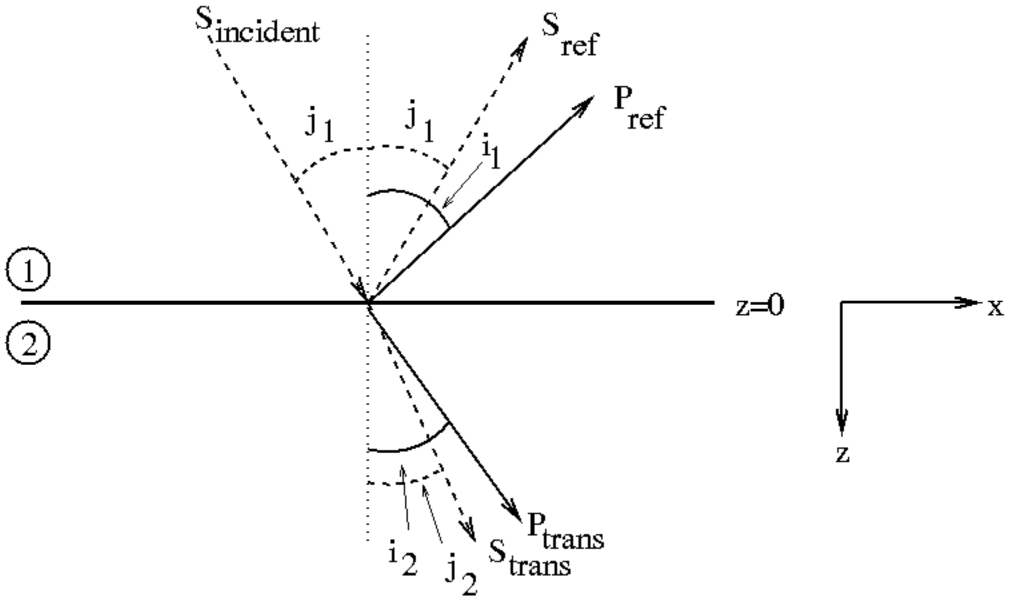

Figure 4: An Incident S wave and Converted Waves, P-S_V System

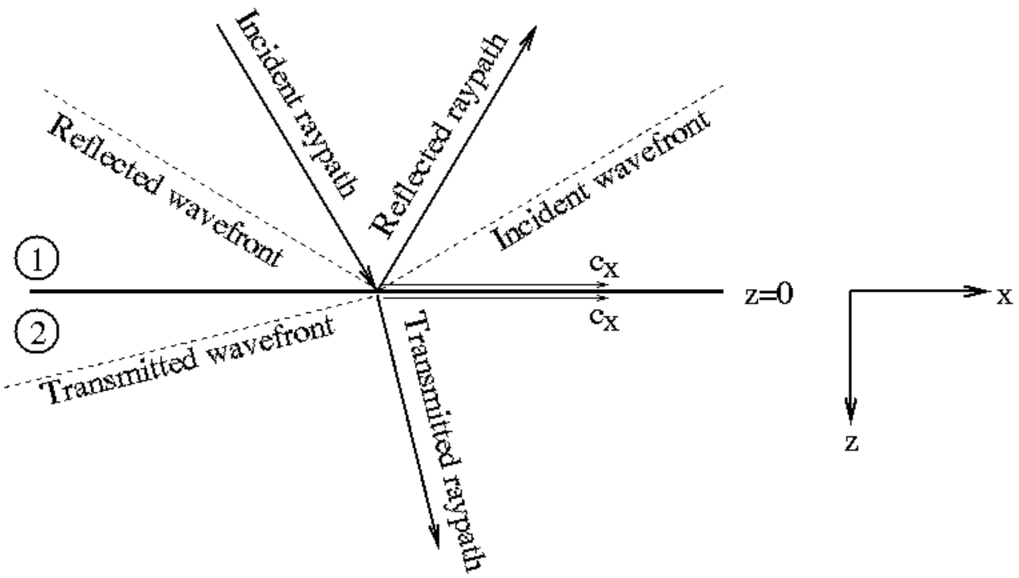

Figure 5: Incident and Converted Plane Wave Ray Paths and Wavefronts

which is Snell's Law. Snell's law indicates that rays in layered media will get bent away from the normal when entering a higher-velocity (stiffer and/or less dense) medium, and will be bent towards the normal when entering a lower-velocity (less stiff and/or denser) medium. An important statement of Snell's Law is that

$$p = \frac{1}{c_x} = \frac{\sin i}{\alpha} = \frac{\sin j}{\beta} \tag{14}$$

is conserved in a system of coupled P and S waves in layered media. p, the horizontal slowness, is commonly referred to as the *ray parameter*.

Because P waves travel faster than S waves, note that when $\alpha_2 > \alpha_1$ in the incident P wave system, there will be *critical angles* of incidence, beyond which the Snell's law angle of incidence in the second medium will not have a real solution. Physically, we shall see that there will be no transmitted plane P wave into the second medium when this occurs. The critical angle for a P-to-P transmitted wave occurs for angles beyond i_{1cPP}, when

$$i_2 = 90^\circ = \sin^{-1}\left(\frac{\alpha_2}{\alpha_1}\sin i_{1cPP}\right) \tag{15}$$

or

$$i_{1cPP} = \sin^{-1}\left(\frac{\alpha_1}{\alpha_2}\right) \; . \tag{16}$$

Similarly, if $\beta_2 > \alpha_1$, there will be another critical angle beyond which there will be no transmitted S wave

$$i_{1cPS} = \sin^{-1}\left(\frac{\alpha_1}{\beta_2}\right) \; . \tag{17}$$

Similarly, if $\beta_2 > \beta_1$, there will be a critical incident angle beyond which there will be no transmitted S wave from an incident S wave

$$j_{1cSS} = \sin^{-1}\left(\frac{\beta_1}{\beta_2}\right) \; . \tag{18}$$

and if $\beta_2 > \alpha_1$, there will be a critical incident angle beyond which there will be no transmitted P wave from an incident S wave

$$j_{1cSP} = \sin^{-1}\left(\frac{\alpha_1}{\beta_2}\right) \; . \tag{19}$$

Plane shear waves propagating in the (x, z) plane with displacements in the $\hat{y}$ direction (Figure 6) will *not* couple with plane waves in the $\mathrm{P} - \mathrm{S_V}$ system for elastically isotropic media. Plane waves in this *S_H system* can be thus treated separately from plane waves in the *$P - S_V$ system*, where in the $\mathrm{S_H}$ system we need only concern ourselves with an incident wave, one reflected wave, and one, transmitted wave. There will be a critical angle in this system for waves impinging on the interface from medium 1 if $\beta_2 > \beta_1$

$$j_{1cSS} = \sin^{-1}\left(\frac{\beta_1}{\beta_2}\right) \; . \tag{20}$$

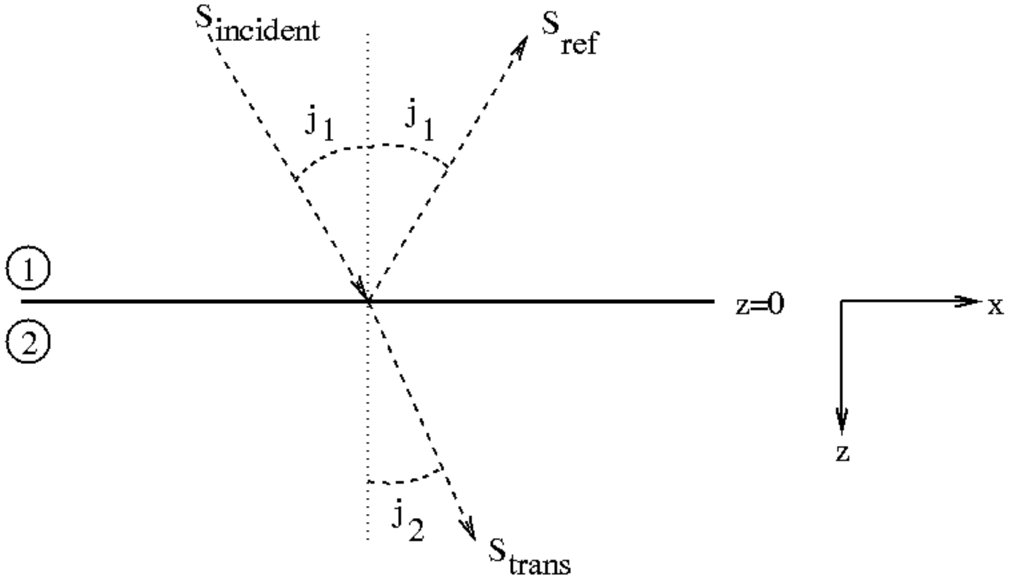

Figure 6: An Incident S wave and Converted Waves, $\mathrm{S_H}$ System

Snell's law dictates how ray path directions change in layered media. To determine how amplitudes are affected, we must match boundary conditions across interfaces (analogously to our development for reflected and transmitted waves on a heterogeneous tensioned string).

We will start with the simpler $\mathrm{S_H}$ system, where there is only one reflected and one transmitted wave to consider.

For harmonic waves in the $\mathrm{S_H}$ system, we write displacement expressions for the incident

$$B_1 e^{\imath(\omega t - k_x x - k_x r_{\beta_1} z)} \tag{21}$$

reflected

$$B_2 e^{\imath(\omega t - k_x x + k_x r_{\beta_1} z)} \tag{22}$$

and transmitted

$$B_3 e^{\imath(\omega t - k_x x - k_x r_{\beta_2} z)} \tag{23}$$

components of the wavefield. where

$$r_{\beta_1} = \cot j_1 = \frac{k_{z,\beta_1}}{k_x} \tag{24}$$

and

$$r_{\beta_2} = \cot j_2 = \frac{k_{z,\beta_2}}{k_x}\ . \tag{25}$$

The welded interface at $z = 0$ demands continuity of displacement there, so that

$$(B_1 + B_2)e^{\imath(\omega t - k_x x)} = (B_3)e^{\imath(\omega t - k_x x)} \tag{26}$$

or

$$B_1 + B_2 = B_3\ . \tag{27}$$

The second continuity conditions is that tractions across the interface $\boldsymbol{\sigma} \cdot \hat{z}$ be continuous at z=0.

For the $\mathrm{S_H}$ wavefield, $u_x = u_z = 0$, so that

$$\boldsymbol{\epsilon} = \begin{pmatrix} 0 & \frac{1}{2}\frac{\partial u_y}{\partial x} & 0 \\ \frac{1}{2}\frac{\partial u_y}{\partial x} & 0 & \frac{1}{2}\frac{\partial u_y}{\partial z} \\ 0 & \frac{1}{2}\frac{\partial u_y}{\partial z} & 0 \end{pmatrix} \tag{28}$$

For isotropic media, we have

$$\sigma_{ij} = \lambda \Theta \delta_{ij} + 2\mu \epsilon_{ij} \tag{29}$$

so that

$$\boldsymbol{\sigma} = \begin{pmatrix} 0 & \mu\frac{\partial u_y}{\partial x} & 0 \\ \mu\frac{\partial u_y}{\partial x} & 0 & \mu\frac{\partial u_y}{\partial z} \\ 0 & \mu\frac{\partial u_y}{\partial z} & 0 \end{pmatrix} \tag{30}$$

The traction at the interface is thus

$$\tau(\hat{z}) = \boldsymbol{\sigma} \cdot \hat{z} = \begin{pmatrix} 0 \\ \mu\frac{\partial u_y}{\partial x} \\ 0 \end{pmatrix} \tag{31}$$

so traction continuity implies that

$$\sigma_{yz}(0^+) = \sigma_{yz}(0^-) \tag{32}$$

or

$$\mu_1 \frac{\partial u_y}{\partial z}(0^+) = \mu_2 \frac{\partial u_y}{\partial z}(0^-)\ . \tag{33}$$

Differentiating the wavefield expressions with respect to z gives

$$\mu_1 \imath k_x r_{\beta_1} B_1 e^{\imath(\omega t - k_x x)} - \mu_1 \imath k_x r_{\beta_1} B_2 e^{\imath(\omega t - k_x x)} = \mu_2 \imath k_x r_{\beta_2} B_3 e^{\imath(\omega t - k_x x)} \tag{34}$$

or

$$\mu_1 k_x r_{\beta_1}(B_1 - B_2) = \mu_2 k_x r_{\beta_2} B_3\ . \tag{35}$$

We can now solve for reflection ($R_{12} = B_2/B_1$) and transmission ($T_{12} = B_3/B_1$) coefficients for the $\mathrm{S_H}$ system

$$\begin{pmatrix} 1 & -1 \\ 1 & \frac{\mu_2 r_{\beta_2}}{\mu_1 r_{\beta_1}} \end{pmatrix} \begin{pmatrix} R_{12} \\ T_{12} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{36}$$

which has solutions

$$R_{12} = \frac{\mu_1 r_{\beta_1} - \mu_2 r_{\beta_2}}{\mu_1 r_{\beta_1} + \mu_2 r_{\beta_2}} \tag{37}$$

and

$$T_{12} = \frac{2\mu_1 r_{\beta_1}}{\mu_1 r_{\beta_1} + \mu_2 r_{\beta_2}} \ . \tag{38}$$

Writing these expressions in terms of the reflection and transmission coefficients gives

$$R_{12} = \frac{\mu_1 \cot j_1 - \mu_2 \cot j_2}{\mu_1 \cot j_1 + \mu_2 \cot j_2} \tag{39}$$

and

$$T_{12} = \frac{2\mu_1 \cot j_1}{\mu_1 \cot j_1 + \mu_2 \cot j_2} \tag{40}$$

where the incidence angles j_1 and j_2 are linked by Snell's Law

$$\sin j_2 = \frac{\beta_2}{\beta_1} \sin j_1 \ . \tag{41}$$

The *vertical slowness* for a wave traveling with velocity c, ray parameter p, and an incidence angle of i is

$$\eta = \sqrt{\frac{1}{c^2} - p^2} = \frac{1}{c}\sqrt{1 - \frac{c^2}{c_x^2}} = \frac{1}{c}\sqrt{1 - \sin^2 i} = \frac{\cos i}{c} \tag{42}$$

$$= \frac{1}{c_x}\sqrt{\frac{c_x^2}{c^2} - 1} = \frac{1}{c_x}\sqrt{\csc^2 i - 1} = \frac{\cot i}{c_x} = \frac{r_{(\alpha,\beta)}}{c_x}$$

which enables us to rewrite the reflection and transmission coefficients as

$$R_{12} = \frac{\mu_1 \eta_{\beta_1} - \mu_2 \eta_{\beta_2}}{\mu_1 \eta_{\beta_1} + \mu_2 \eta_{\beta_2}} \tag{43}$$

and

$$T_{12} = \frac{2\mu_1 \eta_{\beta_1}}{\mu_1 \eta_{\beta_1} + \mu_2 \eta_{\beta_2}} \ . \tag{44}$$

Because $\mu = \rho\beta^2$ and $\eta = \cos i/\beta_i$, we have

$$R_{12} = \frac{\rho_1 \beta_1 \cos j_1 - \rho_2 \beta_2 \cos j_2}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2} \tag{45}$$

and

$$T_{12} = \frac{2\rho_1 \beta_1 \cos j_1}{\rho_1 \beta_1 \cos j_1 + \rho_2 \beta_2 \cos j_2} \ . \tag{46}$$

At vertical incidence, these expressions become

$$R_{12} = \frac{\rho_1 \beta_1 - \rho_2 \beta_2}{\rho_1 \beta_1 + \rho_2 \beta_2} \tag{47}$$

and

$$T_{12} = \frac{2\rho_1\beta_1}{\rho_1\beta_1 + \rho_2\beta_2} \ , \tag{48}$$

which are analogous to the coefficient expressions for the string system (where, recall that each term was also a density-velocity product acoustic impedance).

We can calculate seismic wave energy fluxes by noting that the total energy flux per unit area for a shear plane wavefront is

$$\dot{E} = \frac{\beta B^2 \rho \omega^2}{2} \ . \tag{49}$$

The energy flux incident upon the interface is proportional to the cosine of the incident angle

$$\dot{E}_I = \frac{\beta_1 B_1^2 \rho_1 \omega^2}{2} \cos j_1 \tag{50}$$

and the reflected and transmitted fluxes are, similarly

$$\dot{E}_R = R_{12}^2 \frac{\beta_1 B_1^2 \rho_1 \omega^2}{2} \cos j_1 \tag{51}$$

and

$$\dot{E}_T = T_{12}^2 \frac{\beta_2 B_1^2 \rho_2 \omega^2}{2} \cos j_2 \ . \tag{52}$$

Summing the reflected and transmitted energy fluxes, we have

$$\begin{aligned} &\dot{E}_R + \dot{E}_T = \\ &\frac{B_1^2\omega^2}{2} \cdot \left(\beta_1\rho_1 \cos j_1 \left(\frac{\rho_1\beta_1\cos j_1 - \rho_2\beta_2\cos j_2}{\rho_1\beta_1\cos j_1 + \rho_2\beta_2\cos j_2} \right)^2 + \right. \\ &\left. \beta_2\rho_2\cos j_2 \left(\frac{2\rho_1\beta_1\cos j_1}{\rho_1\beta_1\cos j_1 + \rho_2\beta_2\cos j_2} \right)^2 \right) \\ &= \frac{B_1^2 \cos j_i \beta_1\rho_1\omega^2}{2} \left(\frac{(\rho_1\beta_1\cos j_1 - \rho_2\beta_2\cos j_2)^2 + 4\rho_1\beta_1\rho_2\beta_2\cos j_1\cos j_2}{(\rho_1\beta_1\cos j_1 + \rho_2\beta_2\cos j_2)^2} \right) \end{aligned} \tag{53}$$

The parenthetical term in (53) is unity, so we finally have the desired result:

$$\dot{E}_R + \dot{E}_T = \frac{B_1^2 \cos j_i \beta_1 \rho_1 \omega^2}{2} = \dot{E}_I \ . \tag{54}$$

This is, of course, what we expect for a passive interface that doesn't introduce any new energy to the system, but just redistributes the incident energy flux into the reflected and transmitted waves.

Post-critical $\mathrm{S_H}$ *waves.* We can examine what happens to harmonic plane waves at post-critical incidence in the $\mathrm{S_H}$ system by considering the case where $c_x < \beta_2$. In this case, the disturbance in medium 2 is of the form

$$u_{y_2} = B_3 e^{\imath(\omega t - k_x x - k_x r_{\beta_2} z)} \tag{55}$$

where

$$r_{\beta_2} = c_x\sqrt{\frac{1}{\beta_2^2} - \frac{1}{c_x^2}} = \sqrt{\frac{c_x^2}{\beta_2^2} - 1}\ . \tag{56}$$

When $c_x < \beta_2$ the cotangent of j_2 is imaginary

$$r_{\beta_2} = \pm\imath\sqrt{1 - \frac{c_x^2}{\beta_2^2}}\ , \tag{57}$$

so that (55) can be written as

$$u_{y_2} = B_3 e^{\imath(\omega t - k_x x)} \cdot e^{\mp k_x\sqrt{1-\frac{c_x^2}{\beta_2^2}}\,z}\ . \tag{58}$$

If we chose the physically plausible solution (the $-\imath$ root) which has zero displacement at $z = \infty$, we have a disturbance in medium 2 that decays exponentially in amplitude with increasing z, but is still harmonic (and thus keeps in phase-step as expected with the plane waves in medium 1) in x and t. This type of displacement field is called an *evanescent wave* or *inhomogeneous wave.* Such waves are bound to the interface, and are *not* plane waves. We shall see later that they have a fundamental relationship with *surface waves* and, more generally, other types of *boundary waves.*

The reflection coefficient for the post-critically reflected plane $\mathrm{S_H}$ wave in medium 1 is

$$R_{12} = \frac{\mu_1 r_{\beta_1} + i\mu_2\sqrt{1 - \frac{c_x^2}{\beta_2^2}}}{\mu_1 r_{\beta_1} - i\mu_2\sqrt{1 - \frac{c_x^2}{\beta_2^2}}} \tag{59}$$

which has a magnitude of 1 and introduces a complex phase shift of 2θ, where

$$\theta = \tan^{-1}\frac{\mu_2\sqrt{1 - \frac{c_x^2}{\beta_2^2}}}{\mu_1\sqrt{\frac{c_x^2}{\beta_1^2} - 1}}\ . \tag{60}$$

The transmission coefficient for the post-critical system is

$$T_{12} = \frac{2\mu_1 r_{\beta_1}}{\mu_1 r_{\beta_1} - i\mu_2\sqrt{1 - \frac{c_x^2}{\beta_2^2}}} \tag{61}$$

which controls the amplitude of the evanescent disturbance. There is no problem with energy conservation, as the evanescent wave, once excited by our incident plane wave, does not propagate a net energy flux into medium 2; it gives back as much as it receives.

Let us investigate the $\mathrm{S_H}$ system wave propagation system further by considering another important boundary condition, that of a *free surface* bounded by (effectively) a vacuum. A homogeneous medium bounded by a plane free surface is referred to as a *homogeneous half-space.* The vacuum (or practically

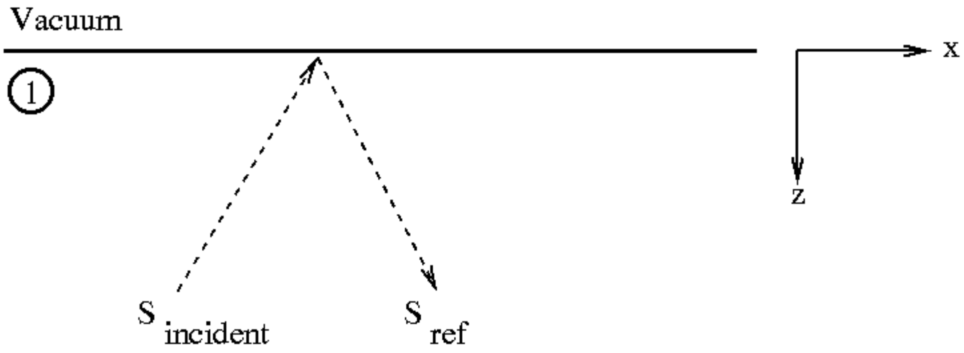

Figure 7: $\mathrm{S_H}$ Wave Incident on Free Surface

speaking, air, which has a low enough acoustic impedance to usually effectively act as a vacuum) applies no tractions to the medium and is stress-free, so the boundary conditions are

$$\tau(\hat{z}) = \boldsymbol{\sigma} \cdot \hat{z} = 0 \ . \tag{62}$$

With a single medium bounded on top by a plane free surface (Figure 7), we clearly will have just an incident and a reflected wave to consider. The sum of the two tractions from these waves must be zero. We showed earlier that the traction on the interface from an $\mathrm{S_H}$ wave is just $\mu(\partial u_y/\partial z)$, so this gives the (almost) trivial result that $\mu r_\beta(B_1 - B_2) = 0$ or just $B_1 = B_2$. So the reflection coefficient is just 1 and the wavefield is defined by the superposition of the incident

$$B_1 e^{\imath(\omega t - k_x x + k_x r_\beta z)} \tag{63}$$

and reflected

$$B_1 e^{\imath(\omega t - k_x x - k_x r_\beta z)} \tag{64}$$

component plane waves. An interesting result is that the disturbance at $z = 0$ is

$$2B_1 e^{\imath(\omega t - k_x x)} \tag{65}$$

which has twice the amplitude of the incident disturbance in the medium. This effect is called the *free surface amplification.*

Next, consider the more complicated situation of a P wave incident on a free surface (Figure 8). The appropriate plane-wave potentials field consists of an incident P wave potential

$$\Phi_I = A_1 e^{\imath(\omega t - k_x x + k_x r_\alpha z)} \ , \tag{66}$$

a reflected P wave potential

$$\Phi_R = A_2 e^{\imath(\omega t - k_x x - k_x r_\alpha z)} \ , \tag{67}$$

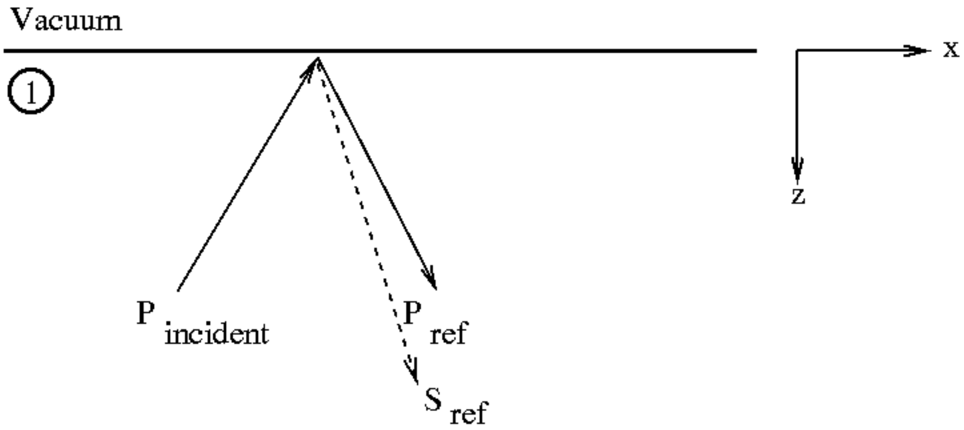

Figure 8: P Wave Incident on Free Surface

and a reflected S wave potential

$$\mathbf{\Psi}_R = B_2 e^{\imath(\omega t - k_x x - k_x r_\beta z)} \hat{y} = \Psi_y \hat{y} \ . \tag{68}$$

The displacements are given by

$$u_x = (\triangledown \Phi + \triangledown \times \mathbf{\Psi})_x = \frac{\partial \Phi}{\partial x} - \frac{\partial \Psi_y}{\partial z} \tag{69}$$

and

$$u_z = (\triangledown \Phi + \triangledown \times \mathbf{\Psi})_z = \frac{\partial \Phi}{\partial z} + \frac{\partial \Psi_y}{\partial x} \ . \tag{70}$$

All stresses and strains are in the (x, z) plane, and the stress tensor for the isotropic medium has the form

$$\boldsymbol{\sigma} = \begin{pmatrix} \lambda\Theta + 2\mu\epsilon_{xx} & 0 & 2\mu\epsilon_{xz} \\ 0 & 0 & 0 \\ 2\mu\epsilon_{xz} & 0 & \lambda\Theta + 2\mu\epsilon_{zz} \end{pmatrix} \tag{71}$$

where

$$\epsilon_{xz} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2}\left(\frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Psi_y}{\partial z^2} + \frac{\partial^2 \Phi}{\partial z \partial x} + \frac{\partial^2 \Psi_y}{\partial x^2}\right) \tag{72}$$

$$= \frac{1}{2}\left(2\frac{\partial^2 \Phi}{\partial x \partial z} - \frac{\partial^2 \Psi_y}{\partial z^2} + \frac{\partial^2 \Psi_y}{\partial x^2}\right)$$

and

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} = \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Psi_y}{\partial x \partial z} \ . \tag{73}$$

At $z = 0$, the ϵ_{xz} and ϵ_{zz} strains (which are the ones that contribute to $\tau(\hat{z}) = \boldsymbol{sigma}\hat{(}z)$) are thus

$$\begin{aligned} \epsilon_{xz}\big|_{z=0} &= \frac{1}{2}(2A_1 r_\alpha k_x^2 - 2A_2 r_\alpha k_x^2 + B_2 k_x^2 r_\beta^2 - B_2 k_x^2) e^{\imath(\omega t - k_x x)} \\ &= k_x^2 \left((A_1 - A_2) r_\alpha + \frac{B_2 (r_\beta^2 - 1)}{2} \right) e^{\imath(\omega t - k_x x)} \end{aligned} \tag{74}$$

and

$$\begin{aligned} \epsilon_{zz}\big|_{z=0} &= (-(A_1 + A_2) r_\alpha^2 k_x^2 - B_2 r_\beta k_x^2) e^{\imath(\omega t - k_x x)} \\ &= -k_x^2 ((A_1 + A_2) r_\alpha^2 + B_2 r_\beta) e^{\imath(\omega t - k_x x)} \ . \end{aligned} \tag{75}$$

The corresponding stress terms are

$$\sigma_{xz} = 2\mu \epsilon_{xz} = \mu k_x^2 \left(2(A_1 - A_2) r_\alpha + B_2 (r_\beta^2 - 1) \right) e^{\imath(\omega t - k_x x)} \tag{76}$$

and

$$\sigma_{zz} = 2\mu \epsilon_{zz} + \lambda \Theta = -2\mu k_x^2 ((A_1 + A_2) r_\alpha^2 + B_2 r_\beta) e^{\imath(\omega t - k_x x)} + \lambda \Theta \tag{77}$$

where the dilatation is just

$$\Theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} \tag{78}$$

as the shear wave has zero dilatation. We thus have

$$\Theta = -k_x^2 ((A_1 + A_2) + r_\alpha^2 (A_1 + A_2)) e^{\imath(\omega t - k_x x)} \tag{79}$$

so

$$\sigma_{zz} = k_x^2 (-2\mu ((A_1 + A_2) r_\alpha^2 + B_2 r_\beta) - \lambda (1 + r_\alpha^2)(A_1 + A_2)) e^{\imath(\omega t - k_x x)} \ . \tag{80}$$

For the traction at the free surface to be zero, we must have $\sigma_{xz} = \sigma_{zz} = 0$ at $z = 0$. Normalizing by the incident P amplitude A_1 gives

$$2r_\alpha - 2r_\alpha R_P + (r_\beta^2 - 1) R_S = 0 \tag{81}$$

for the σ_{xz} expression and

$$\lambda(1 + r_\alpha^2) + \lambda(1 + r_\alpha^2) R_P + 2\mu r_\alpha^2 + 2\mu r_\alpha^2 R_P + 2\mu r_\beta R_S = 0 \tag{82}$$

for the σ_{zz} expression, where the reflection coefficients for the potential amplitudes are $R_P = A_2/A_1$ and $R_S = B_2/A_1$. Gathering coefficient terms and writing these two constraint equations as a system for the unknown P and PS reflection coefficients gives

$$\begin{pmatrix} -2r_\alpha & r_\beta^2 - 1 \\ (\lambda + 2\mu) r_\alpha^2 + \lambda & 2\mu r_\beta \end{pmatrix} \begin{pmatrix} R_P \\ R_S \end{pmatrix} = \begin{pmatrix} -2r_\alpha \\ -(\lambda + 2\mu) r_\alpha^2 - \lambda \end{pmatrix} \ . \tag{83}$$

Rewriting this as

$$\begin{pmatrix} 2r_\alpha & 1-r_\beta^2 \\ (\lambda+2\mu)(1+r_\alpha^2)-2\mu & 2\mu r_\beta \end{pmatrix} \begin{pmatrix} R_P \\ R_S \end{pmatrix} = \begin{pmatrix} 2r_\alpha \\ 2\mu-(\lambda+2\mu)(1+r_\alpha^2) \end{pmatrix} \tag{84}$$

and noting that

$$1+r_\alpha^2 = 1+\cot^2 i = \csc^2 i = \left(\frac{c_x}{\alpha}\right)^2 = \frac{c_x^2\rho}{\lambda+2\mu} \tag{85}$$

gives

$$\begin{pmatrix} 2r_\alpha & 1-r_\beta^2 \\ c_x^2\rho-2\mu & 2\mu r_\beta \end{pmatrix} \begin{pmatrix} R_P \\ R_S \end{pmatrix} = \begin{pmatrix} 2r_\alpha \\ 2\mu-c_x^2\rho \end{pmatrix} . \tag{86}$$

Dividing both sides of the $\sigma_{zz}=0$ equation by μ and noting that $\beta^2=\mu/\rho$ gives

$$\begin{pmatrix} 2r_\alpha & 1-r_\beta^2 \\ \frac{c_x^2}{\beta^2}-2 & 2r_\beta \end{pmatrix} \begin{pmatrix} R_P \\ R_S \end{pmatrix} = \begin{pmatrix} 2r_\alpha \\ 2-\frac{c_x^2}{\beta^2} \end{pmatrix} . \tag{87}$$

But $c_x^2/\beta^2-2 = 1+r_\beta^2-2 = r_\beta^2-1$, so we have

$$\begin{pmatrix} 2r_\alpha & 1-r_\beta^2 \\ r_\beta^2-1 & 2r_\beta \end{pmatrix} \begin{pmatrix} R_P \\ R_S \end{pmatrix} = \begin{pmatrix} 2r_\alpha \\ 1-r_\beta^2 \end{pmatrix} . \tag{88}$$

Solving for the PP and PS reflection coefficients gives

$$R_P = \frac{\begin{vmatrix} 2r_\alpha & 1-r_\beta^2 \\ 1-r_\beta^2 & 2r_\beta \end{vmatrix}}{\begin{vmatrix} 2r_\alpha & 1-r_\beta^2 \\ r_\beta^2-1 & 2r_\beta \end{vmatrix}} = \frac{4r_\alpha r_\beta-(1-r_\beta^2)^2}{4r_\alpha r_\beta+(1-r_\beta^2)^2} \tag{89}$$

and

$$R_S = \frac{\begin{vmatrix} 2r_\alpha & 2r_\alpha \\ r_\beta^2-1 & 1-r_\beta^2 \end{vmatrix}}{\begin{vmatrix} 2r_\alpha & 1-r_\beta^2 \\ r_\beta^2-1 & 2r_\beta \end{vmatrix}} \tag{90}$$

$$= \frac{2r_\alpha(1-r_\beta^2-r_\beta^2+1)}{4r_\alpha r_\beta+(1-r_\beta^2)^2} = \frac{4r_\alpha(1-r_\beta^2)}{4r_\alpha r_\beta+(1-r_\beta^2)^2} .$$

We are still not done, however, as R_P and R_S are displacement *potential* reflection coefficients, not displacement reflection coefficients. To find displacement reflection coefficients, we use

$$\mathbf{u}_P = \bigtriangledown\Phi \tag{91}$$

and

$$\mathbf{u}_S = \bigtriangledown\times\mathbf{\Psi} . \tag{92}$$

For the P$_{\mathrm{SV}}$ plane wave system, we thus have the displacement vectors

$$\mathbf{u}_{P_I} = (u_x, u_z)_{I_R} = (-ik_x,\ ik_x r_\alpha)\Phi_I \ , \tag{93}$$

$$\mathbf{u}_{P_R} = (u_x, u_z)_{P_R}, = (-ik_x,\ -ik_x r_\alpha)\Phi_R \ , \tag{94}$$

and

$$\mathbf{u}_{S_R} = (u_x, u_z)_{S_R} = (-ik_x,\ ik_x r_\beta)\Psi_y \ , \tag{95}$$

that produce the displacement amplitude coefficients

$$\frac{|\mathbf{u}_{P_R}|}{|\mathbf{u}_{P_I}|} = \frac{A_2\sqrt{k_x^2 + k_x^2 r_\alpha^2}}{A_1\sqrt{k_x^2 + k_x^2 r_\alpha^2}} = \frac{A_1}{A_2} = R_P \tag{96}$$

and

$$\frac{|\mathbf{u}_{S_R}|}{|\mathbf{u}_{P_I}|} = \frac{B_2\sqrt{k_x^2 + k_x^2 r_\beta^2}}{A_1\sqrt{k_x^2 + k_x^2 r_\alpha^2}} = \frac{k_\beta}{k_\alpha} R_S \tag{97}$$

where $k_\alpha = \omega/\beta$ and $k_\beta = \omega/\alpha$, so that

$$\frac{|\mathbf{u}_{S_R}|}{|\mathbf{u}_{P_I}|} = \frac{\alpha}{\beta} R_S \ . \tag{98}$$

Figure 9 shows the amplitude reflection coefficients for an incident P wave on a free surface for a Poisson solid. Note that there is a 180° phase shift for the reflected P wave and zero amplitude for the S wave at vertical incidence. Thus, an upgoing P wave with an initial compression will be converted to a downgoing P wave with a dilational first motion. Note also that when the sum of the P-wave and S-wave reflected incidence angles, $i + j$, is exactly 90°, we will have complete conversion from P to S (Figure 9, *right*).

The development for an incident S$_{\mathrm{V}}$ wave (Figure 10) is similar, where we now have an incident S wave potential

$$\mathbf{\Psi}_I = B_1 e^{\imath(\omega t - k_x x + k_x r_\beta z)}\hat{y} = \Psi_{y_1}\hat{y} \ . \tag{99}$$

a reflected P wave potential

$$\Phi_R = A_2 e^{\imath(\omega t - k_x x - k_x r_\alpha z)} = \Phi \ , \tag{100}$$

and a reflected S wave potential

$$\mathbf{\Psi}_R = B_2 e^{\imath(\omega t - k_x x - k_x r_\beta z)}\hat{y} = \Psi_{y_2}\hat{y} \ . \tag{101}$$

Letting $\Psi_y = \Psi_{y_1} + \Psi_{y_2}$, the strains at $z = 0$ are now given by (72) and (73) as

$$\begin{aligned} \epsilon_{xz} &= \frac{k_x^2}{2}(-2A_2 r_\alpha + B_1 r_\beta^2 + B_2 r_\beta^2 - B_1 - B_2)e^{\imath(\omega t - k_x x)} \qquad (102) \\ &= k_x^2\left(-A_2 r_\alpha + \frac{B_1}{2}(r_\beta^2 - 1) + \frac{B_2}{2}(r_\beta^2 - 1)\right)e^{\imath(\omega t - k_x x)} \ . \end{aligned}$$

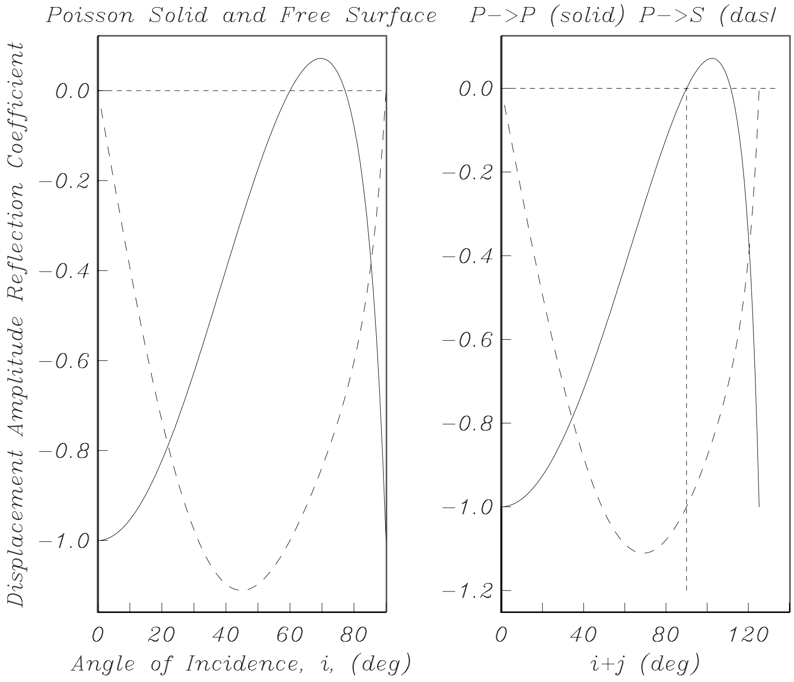

Figure 9: Reflection Coefficients for a P Wave Incident on Free Surface (homogeneous Poisson solid).

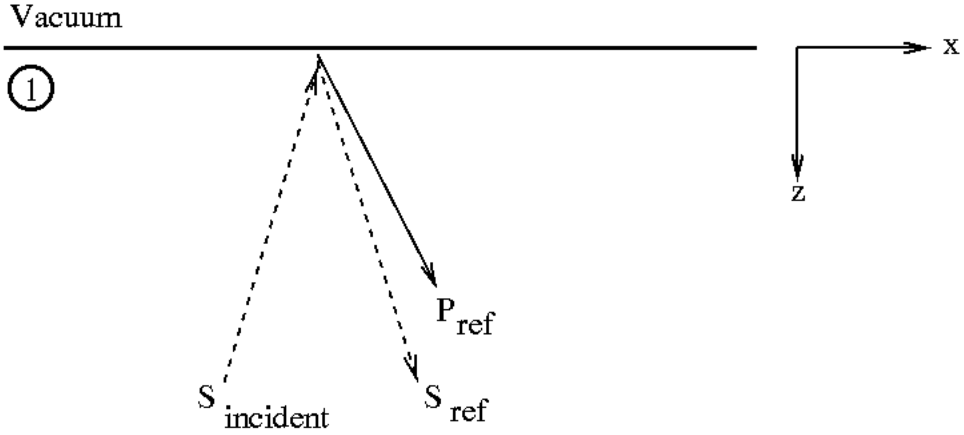

Figure 10: S Wave Incident on Free Surface

and

$$\epsilon_{zz} = k_x^2(-A_2 r_\alpha^2 + B_1 r_\beta - B_2 r_\beta) e^{\imath(\omega t - k_x x)} \ . \tag{103}$$

Applying (76) and (77) to obtain the relevant stress terms at $z = 0$ for the free surface boundary condition gives

$$\sigma_{xz} = 2\mu\epsilon_{xz} = k_x^2 \mu \left(-2A_2 r_\alpha + B_1(r_\beta^2 - 1) + B_2(r_\beta^2 - 1)\right) e^{\imath(\omega t - k_x x)} \tag{104}$$

and

$$\begin{aligned} \sigma_{zz} = 2\mu\epsilon_{zz} + \lambda\Theta &= -2\mu k_x^2 (A_2 r_\alpha^2 - B_1 r_\beta + B_2 r_\beta^2) e^{\imath(\omega t - k_x x)} + k_x^2 \lambda(-A_2(1 + r_\alpha^2) e^{\imath(\omega t - k_x x)} \\ &= k_x^2(-A_2(\lambda(1 + r_\alpha^2) + 2\mu r_\alpha^2) + 2B_1 \mu r_\beta - 2B_2 \mu r_\beta) e^{\imath(\omega t - k_x x)} \ . \end{aligned} \tag{105}$$

Now, setting these two traction components equal to zero and defining the reflection coefficients in this case as $R'_P = A_2/B_1$ and $R'_S = B_2/B_1$, gives the system of equations

$$\begin{pmatrix} -2r_\alpha & r_\beta^2 - 1 \\ (\lambda + 2\mu) r_\alpha^2 + \lambda & 2\mu r_\beta \end{pmatrix} \begin{pmatrix} R'_P \\ R'_S \end{pmatrix} = \begin{pmatrix} 1 - r_\beta^2 \\ 2\mu r_\beta \end{pmatrix} \tag{106}$$

which is the same as (83) except for the right hand side. This makes sense, as the physical system is the same; only the forcing function (an $\mathrm{S_V}$ wave rather than a P wave) has changed. Applying the previous substitutions gives the counterpart of (88)

$$\begin{pmatrix} 2r_\alpha & 1 - r_\beta^2 \\ r_\beta^2 - 1 & 2r_\beta \end{pmatrix} \begin{pmatrix} R'_P \\ R'_S \end{pmatrix} = \begin{pmatrix} r_\beta^2 - 1 \\ 2r_\beta \end{pmatrix} \ . \tag{107}$$

Solving for these displacement potential reflection coefficients gives

$$R'_P = \frac{\begin{vmatrix} r_\beta^2 - 1 & 1 - r_\beta^2 \\ 2r_\beta & 2r_\beta \end{vmatrix}}{\begin{vmatrix} 2r_\alpha & 1 - r_\beta^2 \\ r_\beta^2 - 1 & 2r_\beta \end{vmatrix}} = \frac{4r_\beta(r_\beta^2 - 1)}{4r_\alpha r_\beta + (1 - r_\beta^2)^2} \tag{108}$$

and

$$R'_S = \frac{\begin{vmatrix} 2r_\alpha & r_\beta^2 - 1 \\ r_\beta^2 - 1 & 2r_\beta \end{vmatrix}}{\begin{vmatrix} 2r_\alpha & 1 - r_\beta^2 \\ r_\beta^2 - 1 & 2r_\beta \end{vmatrix}} = \frac{4r_\alpha r_\beta - (1 - r_\beta^2)^2}{4r_\alpha r_\beta + (1 - r_\beta^2)^2} \ . \tag{109}$$

We can convert these to amplitude reflection coefficients as before by multiplying R'_P by β/α (Figure 11).

We see some similar behavior to that observed for the incident P wave, in that complete wave conversion takes place between the incident S and the reflected P when $i + j = 90°$. However, there is a new feature related to the behavior of the coefficients near $j = 35°$, or specifically, where the P-wave reaches an incident angle of $i = 90°$. Snell's law tells us that this occurs at an incident S wave angle of

$$j_c = \sin^{-1}\left(\frac{\beta}{\alpha}\right) \tag{110}$$

which, for a Poisson solid is about 35.26°. For incident S wave angles greater than j_c, we have an evanescent P wave bound to the surface in a situation similar to that described previously for the two-layer cases. $r_\alpha = \cot i$ becomes complex, in this case, and R'_S acquires a magnitude of 1 and a complex phase term that shifts the phase of the reflected S wave. Because this phase shift is independent of frequency, we have the interesting situation that pulse shapes for signals reflecting off of the free surface at angles $j > j_c$ will not preserve their shape, as the relative lag of their frequency components is not preserved on reflection. Generally speaking, whenever an evanescent wave exists, reflection and transmission coefficients will be complex. As before, the evanescent P wave has an amplitude that is nonzero, but energy flux is still conserved, as it does not radiate plane P waves away from the free surface. The evanescent wave decays away from the free surface exponentially with z.

Solving for the general $\mathrm{S_V}$ system for a two layer medium is no different in principle than our past two derivations, but involves a veritable orgy of algebraic manipulation. For a complete set of formulas, see Udías or Aki and Richards. Stacks of layers can also be easily accommodated using *propagator* operators to simulate the net effect of all conversions.

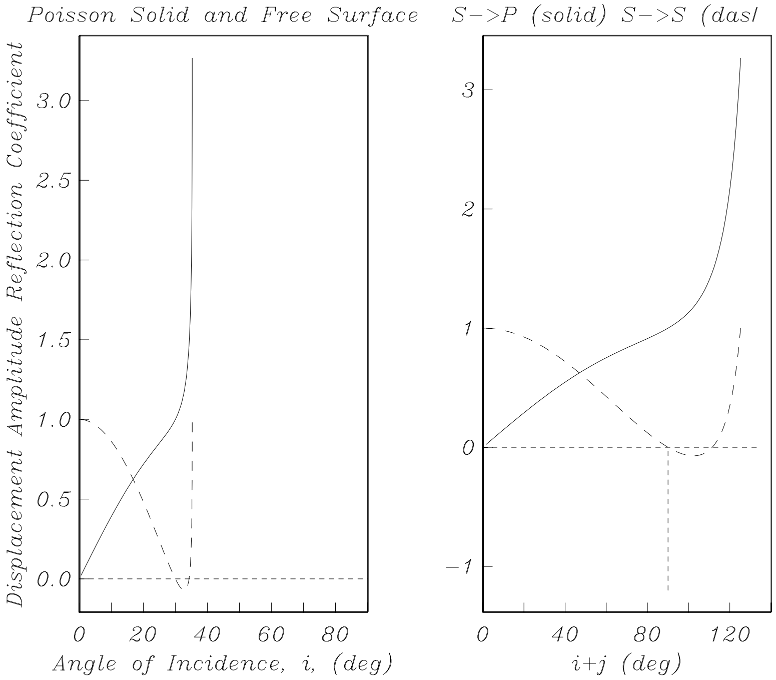

Figure 11: Reflection Coefficients for an S Wave Incident on Free Surface (homogeneous Poisson solid, range of real coefficients). Beyond the critical angle the S to S reflection coefficient becomes unity in magnitude and complex.