

Spring, 2001 Data Processing and Analysis (GEOP 505)

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Poles and Zeros

We showed that for any linear system relating two time functions, $x(t)$ and $y(t)$, the frequency-domain response (the transfer function) can be obtained from the governing linear differential equation with constant coefficients of a single variable

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \dots + b_1 \frac{dx}{dt} + b_0 x . \quad (1)$$

Setting

$$x(t) = e^{i2\pi f t} , \quad (2)$$

$$y(t) = \Phi(f) e^{i2\pi f t} , \quad (3)$$

and solving (1) for $\Phi(f)$ gives the transfer function

$$\Phi(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{j=0}^m b_j (2\pi i f)^j}{\sum_{k=0}^n a_k (2\pi i f)^k} \equiv \frac{Z(f)}{P(f)} \quad (4)$$

where Z and P are complex polynomials in f . The values of f (or equivalently, of the angular frequency $\omega = 2\pi f$) where $Z(f) = 0$ are referred to as *zeros* of (4), as the response of the system will be zero at those frequencies, no matter what the amplitude of the input. Frequencies for which $P(f) = 0$ are referred to as *poles* of (4), as the response of the system will be infinite at those frequencies.

In general, the values of f where we have poles and zeros will be complex. It is useful to express the input function (2) as

$$x(t) = e^{i2\pi f t} = e^{i2\pi(f_r + i f_i)t} = e^{i2\pi f_r t} \cdot e^{-2\pi f_i t} \quad (5)$$

where $f = f_r + i f_i$ and f_r and f_i are real numbers.

This generalized input is:

- A constant for $f = 0$

- A sinusoid for $f_r \neq 0$ and $f_i = 0$.
- A growing exponential for $f_r = 0$ and $f_i < 0$
- A shrinking exponential for $f_r = 0$ and $f_i > 0$
- A growing exponentially weighted sinusoid for $f_r \neq 0$ and $f_i < 0$
- A shrinking exponentially weighted sinusoid for $f_r \neq 0$ and $f_i > 0$

Pole positions are frequently displayed graphically in the complex plane using the convention

$$z \equiv \iota 2\pi f \equiv \sigma + \iota\omega = 2\pi(-f_i + \iota f_r) \quad (6)$$

the positions of the poles in the z plane is especially useful in characterizing the response of the system.

Systems where all pole frequencies have $\sigma < 0$ ($f_i > 0$), so that the poles lie on the left-hand side of the z plane, are stable. In this case the only way to get an infinite output is to drive the system with an exponentially increasing sinusoidal input. The impulse response of such a system will always decay back to zero.

On the other hand, systems where all pole frequencies have $\sigma > 0$ ($f_i < 0$), so that the poles lie on the right-hand side of the z plane, are unstable; we obtain an infinite output even when the input is exponentially decaying. The impulse response of such systems increases in amplitude indefinitely.

Systems where $\sigma = 0$ and $\omega \neq 0$ ($f_i = 0$ and $f_r \neq 0$) have pole frequency responses that are sinusoidal. Such systems will oscillate forever once they get (even marginally) excited at their resonant frequencies.

Figure (1) shows z -pole locations and cartoon impulse responses for various 2-pole system.

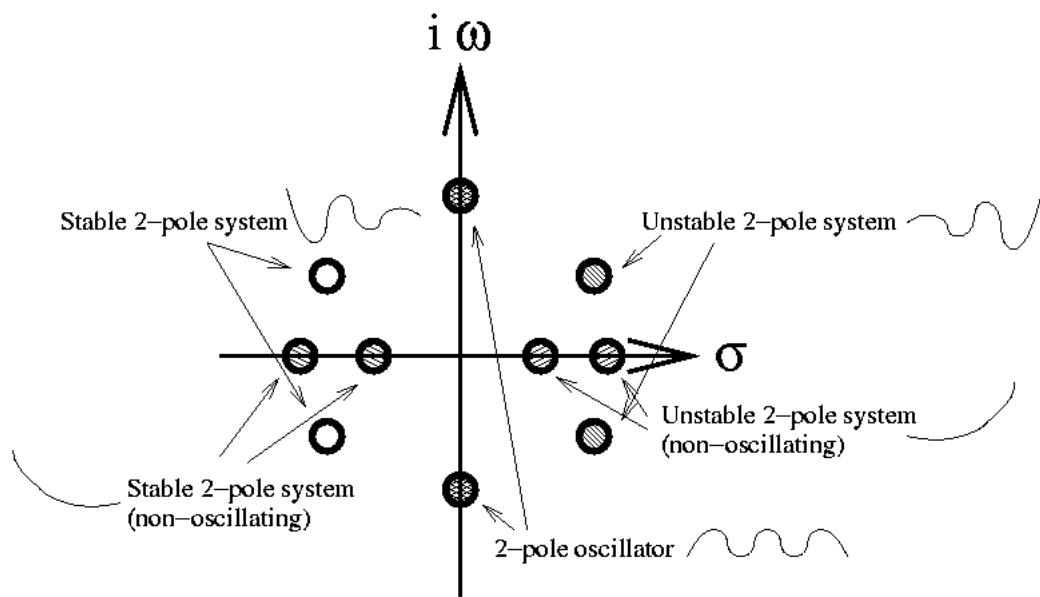


Figure 1: Pole Locations and System Stability