ARMA Processes

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1 Some notation

In the following, we will make use of forward and backward shifts in time. The B operator is defined by

$$Bz_n = z_{n-1}$$

while the F operator is

$$Fz_n = z_{n+1}.$$

Note that B and F are not numbers but rather operators that act on a time series z_n. We can extend this notation to include powers of B and F

$$B^k z_n = z_{n-k}$$

and

$$F^k z_n = z_{n+k}.$$

Also, we can build polynomials from the B and F operators. For example,

$$(1 - B + 0.5B^2)z_n = z_n - z_{n-1} + 0.5z_{n-2}.$$

Another important example is the **backward difference operator**

$$\nabla z_n = (1 - B)z_n = z_n - z_{n-1}.$$

2 White Noise

We will frequently make use of a **white noise**. The white noise process has A_n normally distributed with mean 0, variance σ_A^2, and autocovariance $\gamma_k = 0$, $k = 1, 2, \ldots$ and autocorrelation $\rho_k = 0$, $k = 0, 1, \ldots$. White noise can easily be generated in MATLAB using the **randn** command.

Using our formula for the spectrum of a stationary process from its autocovariance, it's easy to show that the white noise process should have $I(f) = 2\sigma_A^2$, $0 \le f \le 1/2$. In the limit as n goes to infinity, the spectrum is constant for all frequencies. However, for any actual realization of the white noise process, the sample spectrum will contain considerable noise.

3 The ARMA process

An **autoregressive moving average (ARMA)** process is obtained by applying a recursive filter to white noise. In terms of the elements of the z_n and a_n sequences,

$$z_n = \phi_1 z_{n-1} + \phi_2 z_{n-2} + \ldots + \phi_p z_{n-p} + a_n - \theta_1 a_{n-1} - \ldots - \theta_q a_{n-q}.$$

The terms $\phi_1 z_{n-1}$ through $\phi_p z_{n-p}$ are the autoregressive portion of the filter. The terms a_n through $\theta_q a_{n-q}$ are a moving average of the white noise input process. Notice that this has the form of the recursive IIR filter that we previously considered, except that the first coefficients have been normalized to 1.

In terms of our operator notation,

$$\phi(B) z_n = \theta(B) a_n$$

where

$$\phi(B) = 1 - \phi_1 B - \ldots - \phi_p B^p$$

and

$$\theta(B) = 1 - \theta_1 B - \ldots - \phi_q B^q.$$

Note the unusual notational convention of minus signs in front of each coefficient.

Once we've written the filter in our operator notation, it's possible to obtain the transfer function. Let

$$\psi(B) = \frac{\theta(B)}{\phi(B)}.$$

Then

$$z_n = \psi(B) a_n.$$

We can expand $\psi(B)$ as

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots$$

where the coefficients ψ_k can be obtained by Taylor series expansion. Note that the constant coefficient is always 1 and that this time (only) we've used positive signs in front of the coefficients.

An alternative is to let

$$\pi(B) = \psi^{-1}(B) = \frac{\phi(B)}{\theta(B)}.$$

In this case,

$$a_n = \pi(B) z_n.$$

Clearly,

$$\pi(B) = \psi^{-1}(B).$$

We can expand $\pi(B)$ in a Taylor's series as

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \ldots$$

Example 1 Consider the ARMA process

$$Z_n - 0.5Z_{n-1} = A_n - 0.3A_{n-1} + 0.2A_{n-2}.$$

Here $\phi(B) = 1 - 0.5B$ and $\theta(B) = 1 - 0.3B + 0.2B^2$. Using Maple to compute the Taylor's series, we obtain

$$\psi(B) = \frac{\theta(B)}{\phi(B)} = 1 + 0.2B + 0.3B^2 + 0.15B^3 + \ldots$$

Thus $\psi_1 = 0.2$, $\psi_2 = 0.3$, and $\psi_3 = 0.15$. Similarly,

$$\pi(B) = \frac{\phi(B)}{\theta(B)} = 1 - 0.2B - 0.26B^2 - 0.038B^3 - \ldots.$$

Thus $\pi_1 = 0.2$, $\pi_2 = 0.26$, and $\pi_3 = 0.038$.

4 Stationarity and Invertibility

Unfortunately, it is easy to write down an ARMA process which is not stationary. For example, let $Z_n = 2Z_{n-1}$. Then $Var(Z_n) = 4Var(Z_{n-1})$, and the process is clearly not stationary.

It can be shown that if

$$\sum_{j=1}^{\infty} |\psi_j| < \infty$$

then the ARMA process is stationary. This happens if the series $\psi(B)$ converges for every B with $|B| \leq 1$. Since $\psi(B)$ is a rational function, it can also be shown that the series converges for every B with $|B| \leq 1$ if the complex zeros of $\phi(B)$ lie outside the unit circle.

If we have a stationary process, then since $Z_n = \psi(B)A_n$, and the expected values of A_n are all 0, the expected value of Z_n is also 0.

A related issue is that of **invertibility**. Recall that we can write z_n in terms of a_n and previous values of z_{n-k}. That is,

$$\pi(B)z_n = a_n$$

or

$$z_n = a_n + \pi_1 z_{n-1} + \pi_2 z_{n-2} + ...$$

This **inverted form** of the process provides a very useful way of generating a random sequence according to our ARMA process. However, this infinite sum must be truncated in practice. If the π_j coefficients do not decay to zero, then it isn't possible to approximate this infinite sum by truncating it.

We say that the process is **invertible** if

$$\sum_{j=1}^{\infty} |\pi_j| < \infty$$

Since $\pi(B)$ is a rational function, the series is invertible if the complex zeros of $\theta(B)$ lie outside of the unit circle.

Example 2 Recall the ARMA process of example 1. In this case, since $\phi(B) = 1 - 0.5B$, the only zero of $\phi(B)$ is at $B = 2$, which is outside of the unit circle, so the process is stationary. The zeros of $\theta(B)$ are at $B = 0.75 \pm 2.1i$, so the process is also invertible.

5 Finding the autocovariance and autocorrelation of an ARMA process

In order to find the autocovariance of an ARMA process, we start with the model in the recursive filter form.

$$Z_n = \phi_1 Z_{n-1} + \phi_2 Z_{n-2} + \ldots + \phi_p Z_{n-p} + A_n - \theta_1 A_{n-1} - \ldots - \theta_q A_{n-q}.$$

Next, we multiply both sides by z_{n-k} and take expected values. Since $E[Z_n] = E[z_{n-k}] = E[A] = 0$, $Cov(Z_n, Z_{n-k}) = E[Z_n Z_{n-k}]$. Thus

$$\begin{aligned} Cov(Z_n, Z_{n-k}) &= \phi_1 Cov(Z_{n-1}, Z_{n-k}) + \ldots + \phi_p Cov(Z_{n-p}, Z_{n-k}) \\ &\quad + Cov(Z_{n-k}, A_n) - \theta_1 Cov(Z_{n-k}, A_{n-1}) - \ldots - \theta_q Cov(Z_{n-k}, A_{n-q}) \end{aligned}$$

So,

$$\gamma_k = \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p} + \gamma_{ZA}(k) - \theta_1 \gamma_{ZA}(k-1) - \ldots - \theta_q \gamma_{ZA}(k-q)$$

where

$$\gamma_{ZA}(j) = Cov(Z_{n-k}, A_{n-j})$$

Since Z_{n-k} is independent of the white noise at times after $n-k$, these covariances are 0. Also, since $Z_{n-k} = \sum_{j=0}^{\infty} \psi_j A_{n-k-j}$, the remaining covariances are given by $Cov(Z_{n-k}, A_{n-k-j}) = \psi_j \sigma_A^2$. Thus

$$\gamma_{ZA}(j) = \begin{cases} 0 & j > 0 \\ \psi_{-j}\sigma_A^2 & j \leq 0 \end{cases}$$

So, we can express the autocovariance at lag k as

$$\gamma_k = \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p} + \sigma_A^2(-\theta_k \psi_0 - \theta_{k+1}\psi_1 - \ldots - \theta_q \psi_{q-k})$$

When $k \geq q+1$, this simplifies to

$$\gamma_k = \phi_1 \gamma_{k-1} + \ldots + \phi_p \gamma_{k-p}.$$

Another important case is $k = 0$. The variance γ_0 is given by

$$\gamma_0 = \phi_1 \gamma_1 + \ldots + \phi_p \gamma_p + \sigma_A^2(1 - \theta_1 \psi_1 - \ldots - \theta_q \psi_q)$$

These recurrence relations can be solved to obtain the autocovariance and autocorrelation.

The power spectrum can be obtained by substituting $B = e^{-2piif}$ in the transfer function

$$I(f) = 2\sigma_a^2 \left| \frac{\theta(e^{-2\pi if})}{\phi(e^{-2\pi if})} \right|^2 .$$

These computations can all be performed for arbitrary ARMA(p,q) processes. However, in practice, the most important processes have p and q quite small, and general solutions for these particular ARMA(p,q) processes have been developed.

As an example, consider the second order autoregressive process

$$Z_n = \phi_1 Z_{n-1} + \phi_2 Z_{n-2} + A_n$$

It can be show that this process is stationary if $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $-1 < \phi_2 < 1$. Because $\theta(B) = 1$ this process is always invertible.

To compute the autocovariance, we multiply the above formula by Z_{n-k} and take expected values.

$$Cov(Z_n, Z_{n-k}) = \phi_1 Cov(Z_{n-1}, Z_{n-k}) + \phi_2 Cov(Z_{n-2}, Z_{n-k}) + Cov(A_n, Z_{n-k})$$

When $k = 0$, we get

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + Cov(A_n, Z_n)$$

But $Z_n = \phi_1 Z_{n-1} + \phi_2 Z_{n-2} + A_n$, and A_n is independent of Z_{n-1} and Z_{n-2}, so

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + Cov(A_n, A_n)$$

or

$$\gamma_0 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_A^2$$

When $k > 0$, $Cov(A_n, Z_{n-k}) = 0$, and we get

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$$

In terms of the autocorrelation function, we have

$$\rho_0 = 1$$

and

$$\rho_1 = \phi_1 \rho_0 + \phi_2 \rho_1 .$$

Solving this equation for ρ_1, we get

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

For $k > 2$, we get

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \quad k > 2$$

Example 3 Consider an AR(2) process with $\phi_1 = 0.5$ and $\phi_2 = 0.3$. We generated a random sequence according to this process. Figure 1a shows

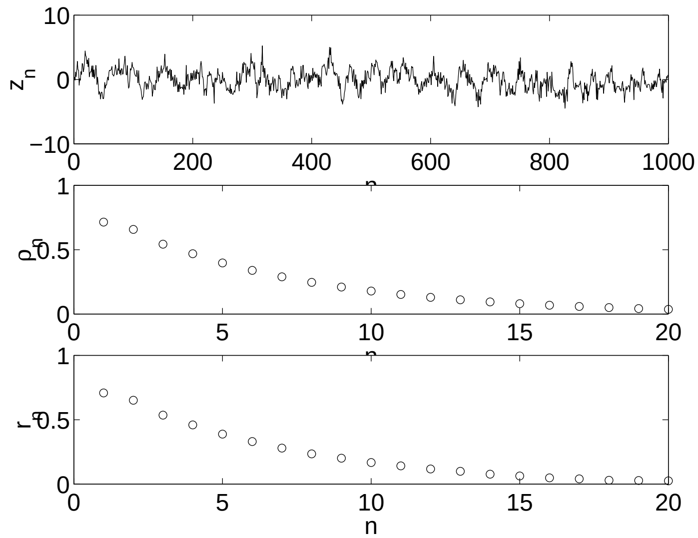

Figure 1: An AR(2) process with $\phi_1 = 0.5$, $\phi_2 = 0.3$.

the first 1000 points of this random process. Figure 1b shows the theoretical autocorrelation. Figure 1c shows the autocorrelation as estimated from the 20,000 point sequence.

Next, consider the ARMA(1,1) process

$$Z_n - \phi_1 Z_{n-1} = A_n - \theta_1 A_{n-1}$$

Here $\phi(B) = 1 - \phi_1 B$ and $\theta(B) = 1 - \theta_1 B$. We need to make sure that the roots of $\phi(B)$ and $\theta(B)$ are outside of the unit circle. This process is stationary if $-1 < \phi_1 < 1$ and invertible when $-1 < \theta_1 < 1$. We can also compute $\psi(B) = 1 + (\phi_1 - \theta_1)B + \ldots$. The recurrence relations for the autocovariance give

$$\gamma_0 = \phi_1 \gamma_1 + \sigma_A^2 (1 - \theta_1 \psi_1)$$

$$\gamma_1 = \phi_1 = \gamma_0 - \theta_1 \sigma_A^2$$

$$\gamma_k = \phi_1 \gamma_{k-1} \;\; k \geq 2$$

These equations can be solved for the autocovariance. We can then convert the solution to an autocorrelation function. The result is

$$\rho_1 = \frac{(1 - \phi_1 \theta_1)(\phi_1 - \theta_1)}{1 + \theta_1^2 - 2\phi_1 \theta_1}$$

$$\rho_k = \phi_1 \rho_{k-1} \;\; k >= 2$$

Box, Jenkins, and Reinsel contains specific solutions for the autocorrelations of a variety of ARMA(p,q) processes with small values of p and q.