Spring, 2008 Theoretical Seismology (GEOP 523)
Homework 3; Due 3/19/08

February 22, 2008

1) For an ideal fluid, the constitutive relation between stress and strain is

$$\sigma_{ij} = \kappa \Theta \delta_{ij} \ . \tag{1}$$

where κ is the incompressibility.

Derive the wave equation for propagating elastic waves in such a fluid by balancing the contact forces and the product of mass and acceleration in this medium. Describe the particle motions for these elastic waves. Show the details of your derivation. What is the velocity of body waves in sea water?

2) Consider an inhomogeneous isotropic medium where the Lamé elastic parameters are functions of position In this case the spatial derivatives of stress found from differenting the constituitive relationship will now include terms that are proportional to spatial derivatives of the elastic moduli.

a) Show, that the appropriate expression for the equation of motion for inhomogeneous isotropic media, i.e., where

$$\sigma_{ij} = \lambda(\mathbf{x})\Theta\delta_{ij} + 2\mu(\mathbf{x})\epsilon_{ij}$$

and $\rho = \rho(\mathbf{x})$ is

$$\rho\frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} = \mu \bigtriangledown^2 u_i + (\lambda+\mu)\frac{\partial \Theta}{\partial x_i} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\partial \mu}{\partial x_j} + \Theta\frac{\partial \lambda}{\partial x_i} \ . \tag{2}$$

Consider the case where the elastic moduli and the density vary exponentially in the $\hat{x_3}$ direction as

$$\lambda(\mathbf{x}) = \lambda_0 e^{\alpha x_3} \tag{3}$$

$$\mu(\mathbf{x}) = \mu_0 e^{\alpha x_3} \tag{4}$$

$$\rho(\mathbf{x}) = \rho_0 e^{\alpha x_3} \ , \tag{5}$$

where α is a positive constant.

For plane body waves propagating in the $\hat{x_3}$ direction with displacements in the $\hat{x_2}$ and $\hat{x_3}$ directions:

b) Verify that your solutions approach the homogeneous medium solutions when the inhomogeneity parameter α approaches zero.

c) Discuss if such plane waves maintain their amplitude with increasing x_3.

d) Discuss effects due to changing the frequency/wavelength of the waves with respect to the scale length α^{-1}.

e) Determine and discuss if waves of arbitrarily long wavelength can propagate in this medium.

Hint: This medium will be dispersive (velocity will be a function of ω). Substitute general P- and S-wave harmonic plane wave displacement solutions propagating in the $\hat{x_3}$ direction,

$$u_3 = Ce^{\imath(\omega t - kx_3)}\hat{x_3} \tag{6}$$

and

$$u_2 = Ce^{\imath(\omega t - kx_3)}\hat{x_2} \tag{7}$$

respectively, and solve for a complex wavenumber, k, which will be a function of ω, α, and the $x_3 = 0$ body wave velocities

$$v_{s0} = \sqrt{\frac{\mu_0}{\rho_0}} \tag{8}$$

$$v_{p0} = \sqrt{\frac{\lambda_0 + 2\mu_0}{\rho_0}} \; . \tag{9}$$